Choosing an allocation in the core.

$(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ a basic EOQ system without holding costs and with transportation costs.

- $\Pi_1(N,\bar{c}^1) = \{\sigma \in \Pi(N) \mid a_i \geq a_j$ implies that $\sigma(i) \leq \sigma(j)$, for all $i,j \in N\}$.

- $\Pi_2(N,\bar{c}^1) = \{\sigma \in \Pi(N) \mid \frac{d_i}{K_i} \geq \frac{d_j}{K_j}$ implies that $\sigma(i) \leq \sigma(j)$, for all $i,j \in N\}$.

The two-line rule

For any $i \in N$,

$$TL_i(N,\bar{c}^1) = \frac{1}{2|\Pi_1(N,\bar{c}^1)|} \sum_{\sigma\in\Pi_1(N,\bar{c}^1)} m_i^\sigma(N,\bar{c}^1) + \frac{1}{2|\Pi_2(N,\bar{c}^1)|} \sum_{\sigma\in\Pi_2(N,\bar{c}^1)} m_i^\sigma(N,\bar{c}^1),$$

where $m_i^\sigma(N,\bar{c}^1) = \bar{c}^1(P_i^\sigma \cup \{i\}) - \bar{c}^1(P_i^\sigma)$ and $P_i^\sigma = \{j \in N \mid \sigma(j) < \sigma(i)\}$.

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Theorem

Let $(N, a, \{a_i\}_{i\in N}, \{d_i\}_{i\in N}, \{K_i\}_{i\in N})$ be a basic EOQ system without holding costs and with transportation costs, and let $(N,\bar{c}^1)$ be its associated cost game. If $(N,\bar{c}^1)$ is subadditive, then $TL(N,\bar{c}^1) \in Core(N,\bar{c}^1)$.

An Economic Shortage Level problem (abbreviated ESL problem) is a deterministic continuous review inventory problem with fixed holding costs and with shortages. The relevant parameters associated to one of these problems are:

- $a > 0$, the fixed cost per order,
- $b_i > 0$, the shortage cost per item and per time unit,
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The objective is to identify the (maximum) shortage level which minimizes the average cost per time unit.

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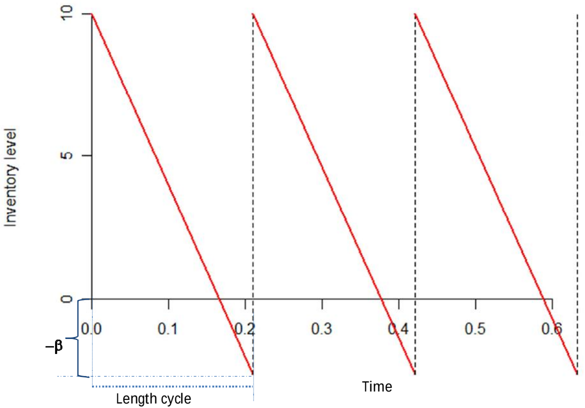
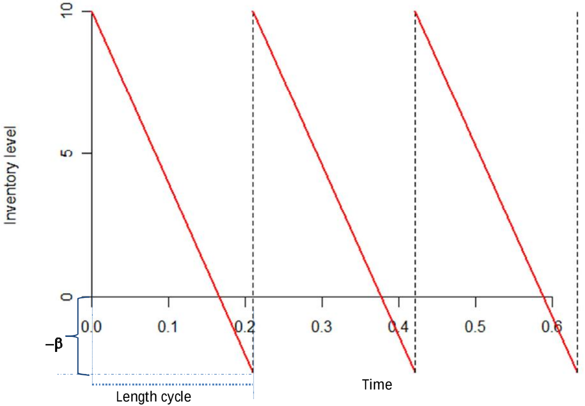
An Economic Shortage Level problem



Identifying the optimal shortage level $\bar{\beta}_i$ is equivalent to identifying the optimal number of orders per time unit $\bar{x}_i = \frac{d_i}{K_i + \bar{\beta}_i}$.



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An Economic Shortage Level problem

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Agent i's cost per time unit

$$C^i(\beta) = \begin{cases} \frac{ad_i}{K_i+\beta_i} & \text{if } -K_i < \beta_i \leq 0 \\ \frac{ad_i}{K_i+\beta_i} + \frac{b_i\beta_i^2}{2(K_i+\beta_i)} & \text{if } \beta_i > 0 \end{cases}$$

where β_i is his shortage level ($\beta_i > -K_i$, $\beta_i > 0$ implies shortage is present).

$$C^i(x_i) = \begin{cases} ax_i & \text{if } x_i \geq \frac{d_i}{K_i} \\ ax_i + \frac{b_i(d_i - K_i x_i)^2}{2x_i d_i} & \text{if } 0 < x_i \leq \frac{d_i}{K_i} \end{cases}$$

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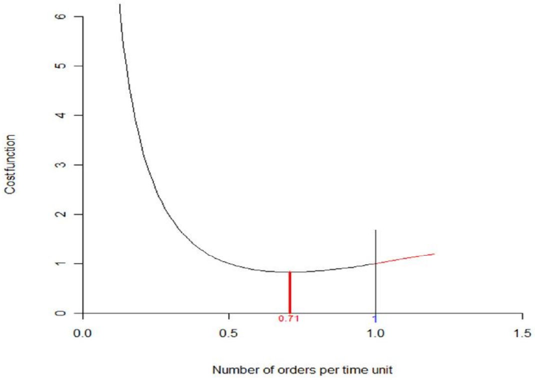

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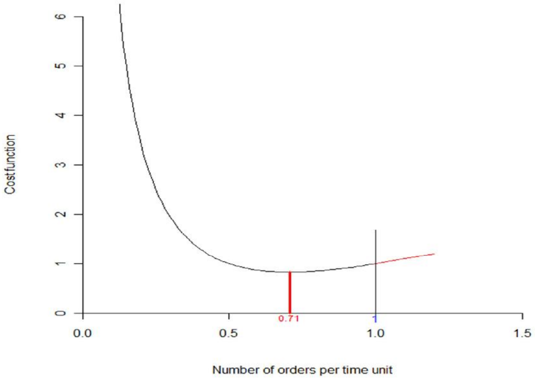

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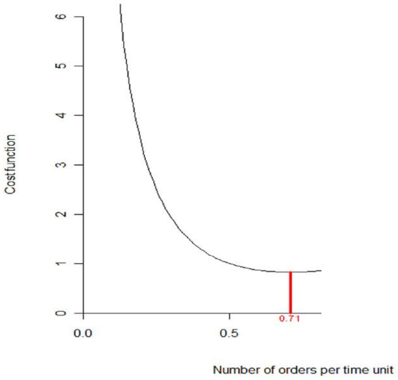

$$x_i^* = \sqrt{\frac{b_i d_i}{2a + b_i \frac{K_i^2}{d_i}}},$$

$$\beta_i^* = -K_i + \frac{d_i}{x_i^*} > 0,$$

$$C^i(x_i^*) = b_i\left[\frac{d_i}{x_i^*} - K_i\right].$$

More than one agent: $N = \{1, \ldots, n\}$, $i \in N$,

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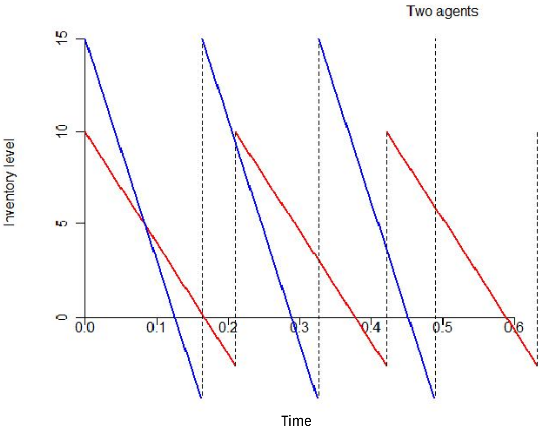

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Agents in $S \subseteq N$ cooperate by placing joint orders

$$\frac{1}{x_i} = \frac{K_i + \beta_i}{d_i} = \frac{K_j + \beta_j}{d_j} = \frac{1}{x_j}, \quad \text{for every } i, j \in S.$$

Cost per time unit for coalition S:

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More than one agent: N
Coalition $S \subset N$
Fix a player $j \in S$ and let $x = x_j$.
Equal length cycle condition implies: $\beta_i = -K_i + \frac{d_i}{x}$ for all $i \in S$.
Thus, for every $x > 0$,

$$C^S(x) = ax + \frac{1}{x} \sum_{i \in S} \frac{b_i}{2d_i} \max{}^2\{-K_i x + d_i, 0\}.$$

$$C^S(x) = ax + \frac{1}{x} \sum_{i \in S_x} \frac{b_i}{2d_i} (-K_i x + d_i)^2$$

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What is the optimal number of orders per time unit for coalition S?

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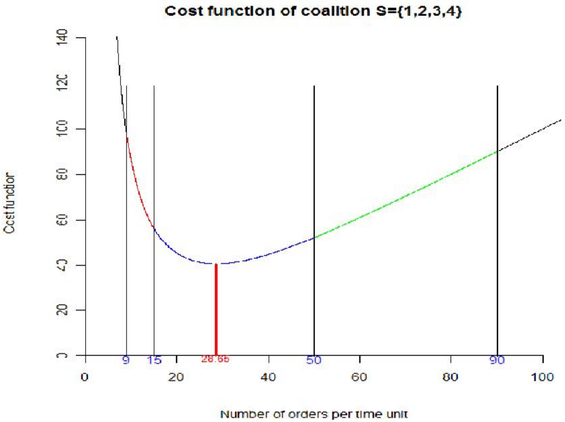

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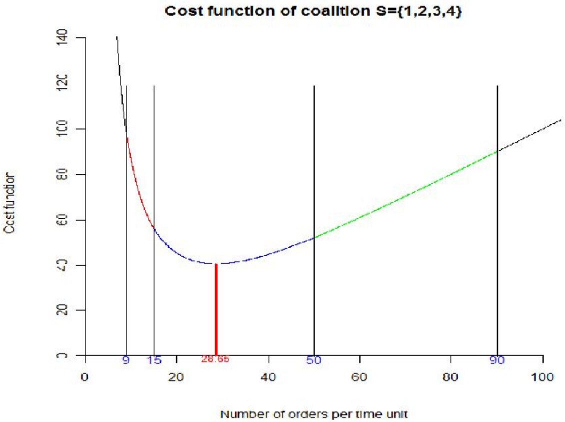

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$$x_S^* = \sqrt{\frac{\sum_{i\in S_{x_S^*}} \frac{b_i d_i}{2}}{a + \sum_{i\in S_{x_S^*}} \frac{b_i}{2}\frac{K_i^2}{d_i}}} = \sqrt{\frac{\sum_{i\in S_{x_S^*}} b_i d_i}{2a + \sum_{i\in S_{x_S^*}} b_i \frac{K_i^2}{d_i}}},$$

with $x_S^* < \frac{d_i}{K_i}$ for all $i \in S_{x_S^*}$

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How do we solve this Equation?

1. Let $S = \{i_1, i_2, \ldots, i_s\}$ be the agents in S arranged in non-decreasing order of the ratios demand/capacity. Thus $\frac{d_{i_1}}{K_{i_1}} \leq \frac{d_{i_2}}{K_{i_2}} \leq \cdots \leq \frac{d_{i_s}}{K_{i_s}}$.
2. Initialize $k = s + 1$, $T = \emptyset$, $x_T = 0$, and $S_{x_T} = S$.
3. Do while $S_{x_T} \neq T$:
   Set $k = k - 1$, $T = T \cup \{i_k\}$, and compute

$$x_T = \sqrt{\frac{\sum_{i_l \in T} b_{i_l} d_{i_l}}{2a + \sum_{i_l \in T} b_{i_l} \frac{K_{i_l}^2}{d_{i_l}}}} \text{ and } S_{x_T} = \{i_l \in S : x_T < \frac{d_{i_l}}{K_{i_l}}\}.$$

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An example

i	d_i	b_i	K_i	$\frac{d_i}{K_i}$
1	0.45	15	5	0.090
2	0.95	15	7.5	0.127
3	1.05	10	8	0.131
4	1.20	12	9	0.133

$a = 180$, compute x_S^* for $S = \{1, 2, 4\}$.

- Take $T = \{4\}$ and compute

$$x_T = \sqrt{\frac{\sum_{i \in T} b_i d_i}{2a + \sum_{i \in T} b_i \frac{K_i^2}{d_i}}} = \sqrt{\frac{b_4 d_4}{2a + b_4 \frac{K_4^2}{d_4}}} = 0.11094.$$

Note that $S_{x_T} = \{i \in S : x_T < \frac{d_i}{K_i}\} = \{2, 4\}$ and $S_{x_T} \neq T$.

- Take $T = \{2, 4\}$ and compute

$$x_T = \sqrt{\frac{\sum_{i \in T} b_i d_i}{2a + \sum_{i \in T} b_i \frac{K_i^2}{d_i}}} = \sqrt{\frac{b_2 d_2 + b_4 d_4}{2a + b_2 \frac{K_2^2}{d_2} + b_4 \frac{K_4^2}{d_4}}} = 0.117983.$$

Now, since $S_{x_T} = \{2, 4\} = T$, $x_S^* = x_T = 0.117983$.