The Core

$(N, c) \in \mathcal{G}$

$x \in \mathbb{R}^n$ belongs to the **core** if and only if

$$x(S) = \sum_{i \in S} x_i \leq c(S), \quad \text{and} \quad x(N) = c(N).$$

$x \in Core(N, v)$

Example 2 (cont.)

$N = \{1, 2, 3\}$, $c(1) = 5$, $c(2) = 5$, $c(3) = 2$, $c(1, 2) = 7$, $c(1, 3) = 7$, $c(2, 3) = 7$, $c(1, 2, 3) = 9$.

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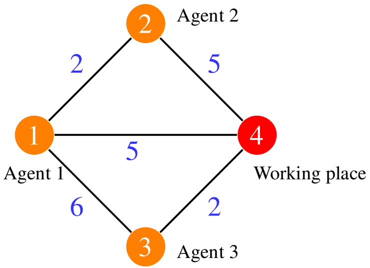

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$Core(N, c) = conv\{(5, 2, 2), (2, 5, 2)\}$

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Some solution concepts

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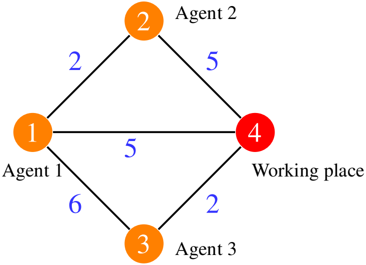

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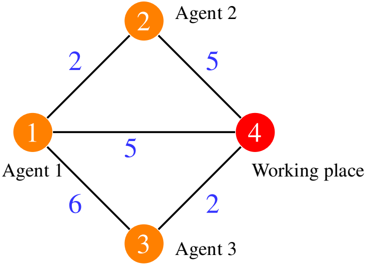

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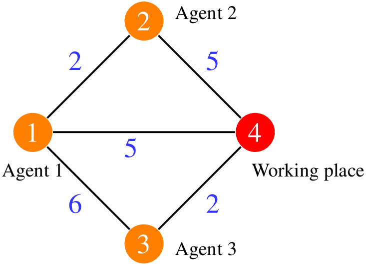

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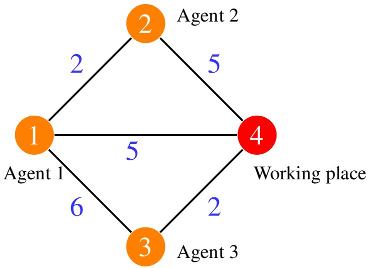

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Example 4 (cont).

$N = \{1, 2\}, \quad c(1) = 1,\ c(2) = 5, c(1, 2) = 7.$

$I(N, c) = \emptyset$ then $Core(N, c) = \emptyset$

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$(N, c) \in G, \quad f$ a value.

Some properties

- **Efficiency:**

$$\sum_{i \in N} f_i(N, c) = c(N).$$

- **Symmetry:**
If $i, j \in N$ with $c(S \cup i) = c(S \cup j)$ for every $S \subset N \setminus \{i, j\}$, then

$$f_i(N, c) = f_j(N, c).$$

- **Dummy:**
If $i \in N$ with $c(S \cup i) = c(S)$ for every $S \subset N \setminus i$, then

$$f_i(N, c) = 0.$$

- **Additivity:**
Take $(N, c) \in G$ and $(N, \bar{c}) \in G$. Then,

$$f_i(N, c) + f_i(N, \bar{c}) = f_i(N, c + \bar{c}), \text{ for every } i \in N$$

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Theorem. (Shapley, 1953)

There is a unique value defined on $\mathcal{G}$ satisfying **Efficiency**, **Symmetry**, **Dummy**, and **Additivity**. It is called **the Shapley value**.

How to compute it?

$(N, c) \in G$, $i \in N$,

$$\Phi_i(N, c) = \sum_{S \subset N \setminus \{i\}} \frac{1}{n\binom{n-1}{|S|}} \left(c(S \cup \{i\}) - c(S) \right)$$

$\Pi(N)$ is the set of orderings of the elements of N.

$$\Phi_i(N, c) = \frac{1}{|\Pi(N)|} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(N, c),$$

where $m_i^{\sigma}(N, c) = c(P_i^{\sigma} \cup \{i\}) - c(P_i^{\sigma})$ and $P_i^{\sigma} = \{j \in N \mid \sigma(j) < \sigma(i)\}$.

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$$f_i(N, c) - f_i(N \setminus j, c_{-j}) = f_j(N, c) - f_j(N \setminus i, c_{-i})$$

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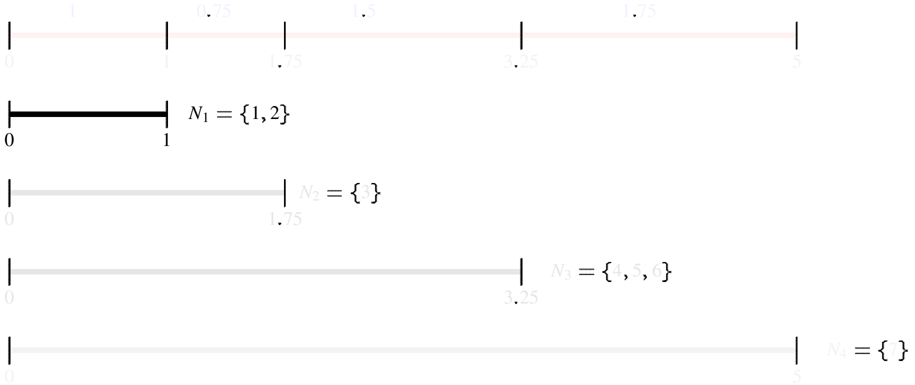
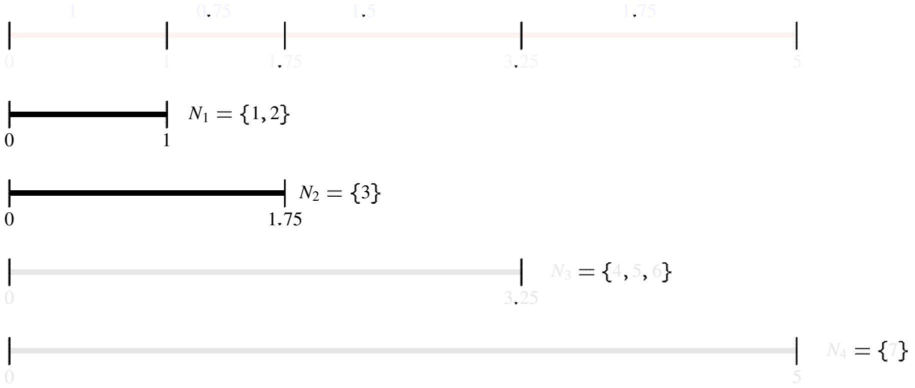


The game.

$N = \{1, 2, 3, 4, 5, 6, 7\} = \cup_{k \in K} N_k$, $K = \{1, 2, 3, 4\}$
$c_1 = c_2 = 1$, $c_3 = 1.75$, $c_4 = c_5 = c_6 = 3.25$, $c_7 = 5$;

For every $S \subset N$:

$$c(S) = \max\{c_k : N_k \cap S \neq \emptyset\}.$$



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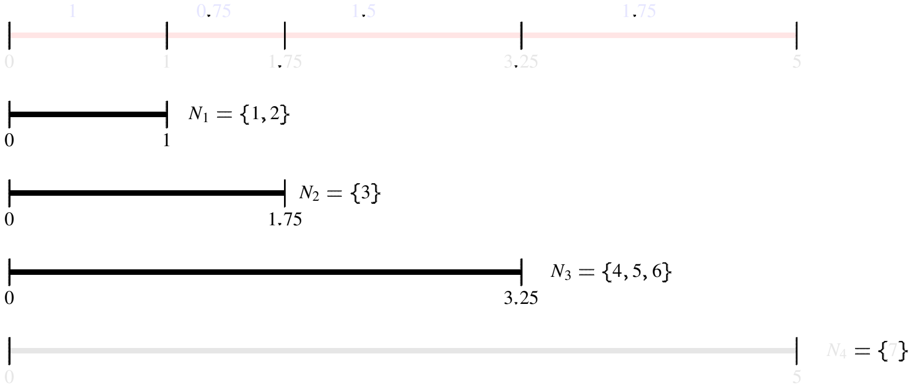

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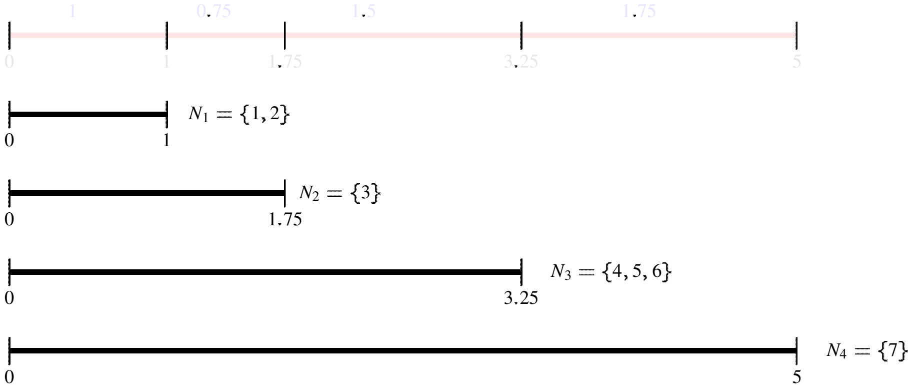

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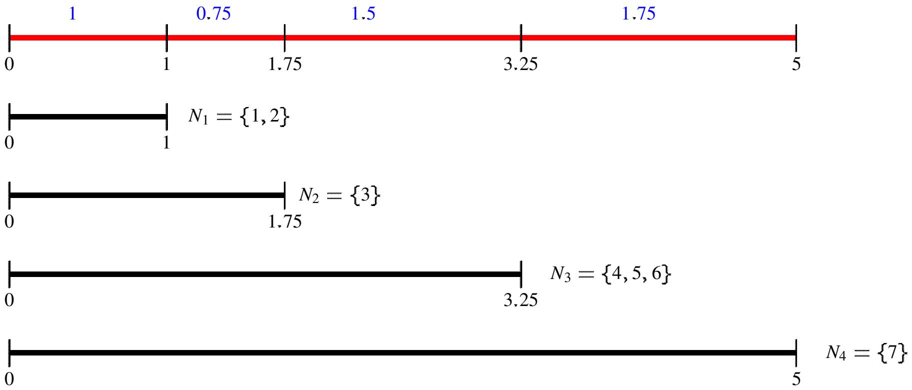

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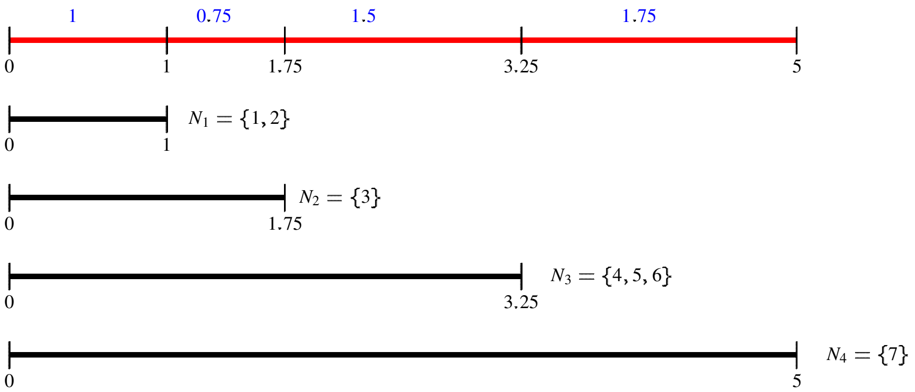

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