MODELLING COOPERATION

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Outline

- **TU Cost Game**: $(N, c) \in \mathcal{G}$
  - A set of agents: $N = \{1, \ldots, n\}$
  - A characteristic function: c

$$\text{For every } S \subset N, \quad c(S) \in \mathbb{R}.$$

Example 1.

$N = \{1, 2\}$,
$c(1) = 6$, $c(2) = 5$, $c(1, 2) = 7$.

- **An allocation**: $x \in \mathbb{R}^n$. $\quad (5, 1), \quad (-1, 10)$
- **An imputation**: $x \in I(N, c)$ if and only if

$$\left.\begin{array}{ll} \textbf{Efficient:} & x(N) = \sum_{i \in N} x_i = c(N), \text{ and} \\ \textbf{Individual rational:} & x_i \leq c(i), \quad \text{for all } i \in N. \end{array}\right\} \quad (3, 4), \quad (4, 3).$$

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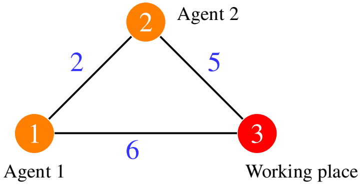

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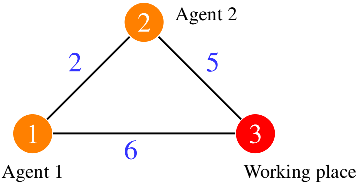

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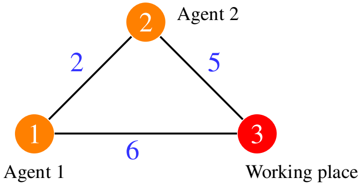

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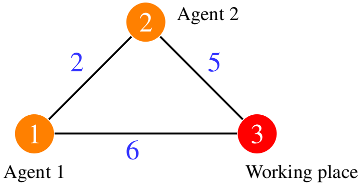

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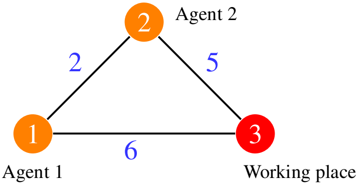

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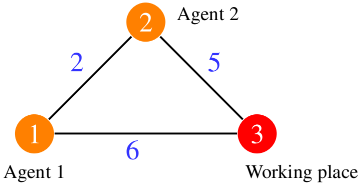

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$(N, c) \in \mathcal{G}$

Some interesting properties:

Subadditivity:

$$c(S \cup T) \leq c(S) + c(T), \quad \text{for every } S, T \subset N,\ S \cap T = \emptyset.$$

Concavity:

$$c(T \cup i) - c(T) \leq c(S \cup i) - c(S), \quad \text{for every } S, T \subset N \setminus i,\ S \subset T.$$

If (N, c) is concave, then (N, c) is subadditive.

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$N = \{1, 2, 3\}, \quad c(1) = 5, c(2) = 5,$

$c(3) = 2, c(1,2) = 7, c(1,3) = 7,$

$c(2,3) = 7, c(1,2,3) = 9.$

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and so on.

This game is concave!

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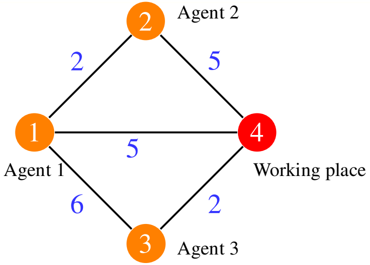

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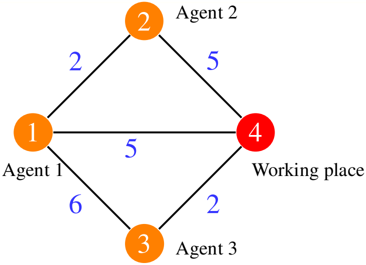

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$$f : (N, c) \in \mathcal{G} \longrightarrow f(N, c) \subset \mathbb{R}^n.$$

$$(N, c) \in \mathcal{G} \longrightarrow I(N, c)$$

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Preliminaries

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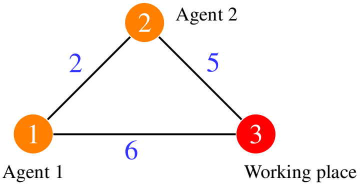

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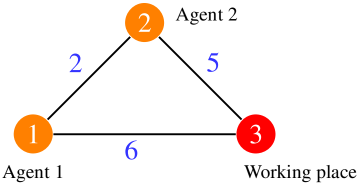

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If $f(N, c)$ always contains a unique element, f is called a **value**. Otherwise, f is a **set-solution** concept.

A value f is **efficient** if and only if $\sum_{i \in N} f_i(N, c) = c(N)$, for all (N, c).

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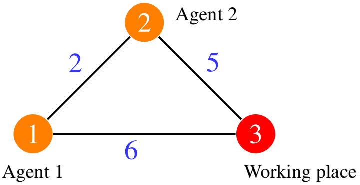

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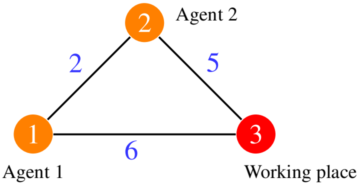

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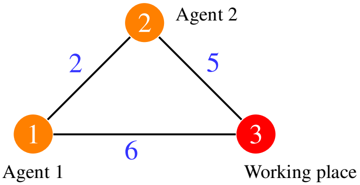

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