

MODELLING COOPERATION

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Outline

Preliminaries

Some solution concepts

A classical example: Airport game

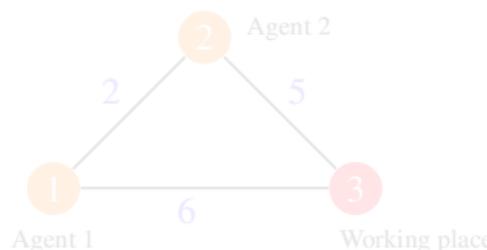
A Linear Programming Game

- **TU Cost Game:** $(N, c) \in \mathcal{G}$

- A set of agents: $N = \{1, \dots, n\}$
- A characteristic function: c

For every $S \subset N$, $c(S) \in \mathbb{R}$.

Example 1.



$$\begin{aligned} N &= \{1, 2\}, \\ c(1) &= 6, c(2) = 5, c(1, 2) = 7. \end{aligned}$$

- An allocation: $x \in \mathbb{R}^n$. $\quad (5, 1), \quad (-1, 10)$
- An imputation: $x \in I(N, c)$ if and only if

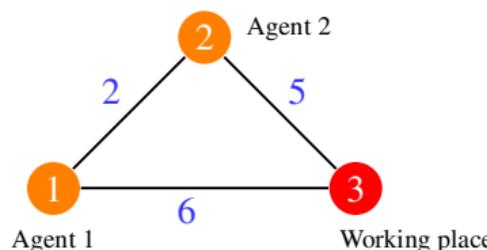
Efficient: $x(N) = \sum_{i \in N} x_i = c(N)$, and
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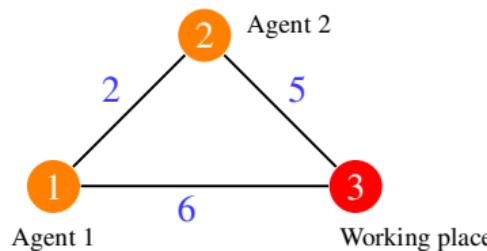
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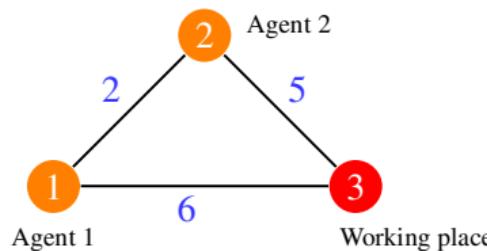
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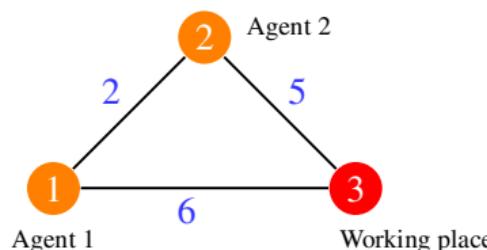
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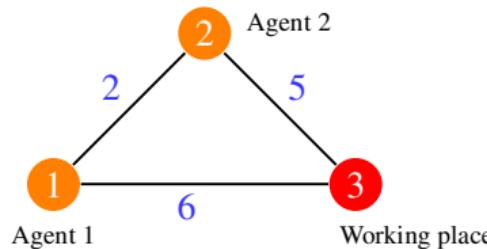
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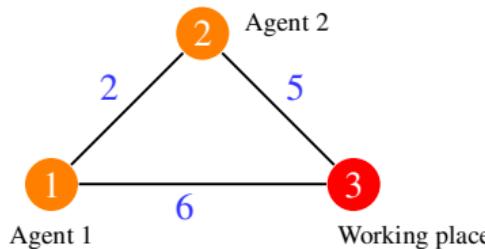
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$$(N, c) \in \mathcal{G}$$

Some interesting properties:

Subadditivity:

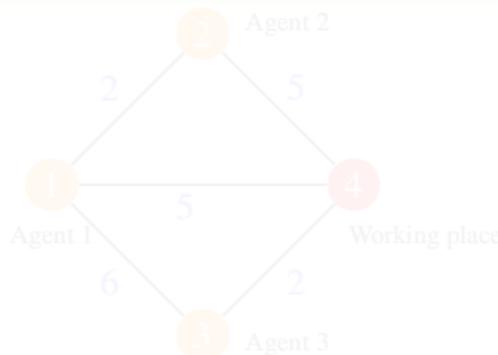
$$c(S \cup T) \leq c(S) + c(T), \quad \text{for every } S, T \subset N, S \cap T = \emptyset.$$

Concavity:

$$c(T \cup i) - c(T) \leq c(S \cup i) - c(S), \quad \text{for every } S, T \subset N \setminus i, S \subset T.$$

If (N, c) is concave, then (N, c) is subadditive.

Example 2.



$$\begin{aligned} N &= \{1, 2, 3\}, \quad c(\emptyset) = 5, c(2) = 5, \\ c(1) &= 2, \quad c(1, 2) = 7, \quad c(1, 2) = 7, \\ c(1, 3) &= 7, \quad c(1, 2, 3) = 9. \\ &= c(1, 2) - c(\emptyset) \leq c(1) - c(\emptyset) = \end{aligned}$$

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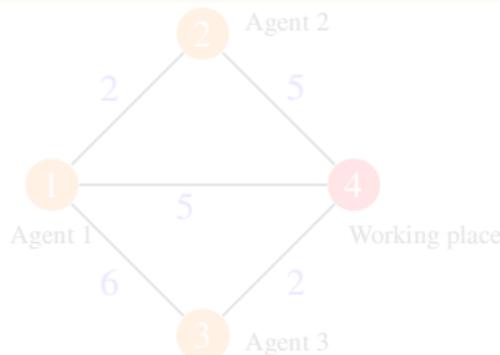
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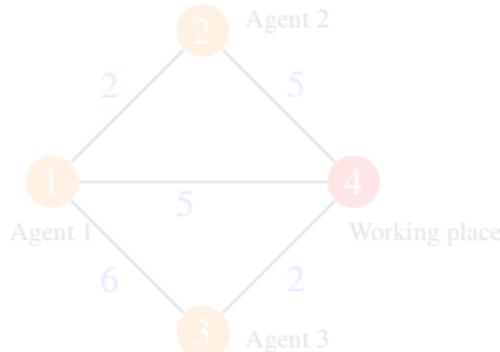
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$$c(2, 3) = 7, \quad c(1, 2, 3) = 9.$$

$$= (1, 2, 3) - (1, 2) \leq (1, 2) - (\emptyset) =$$

and we are

This game is concave!

$$(N, c) \in \mathcal{G}$$

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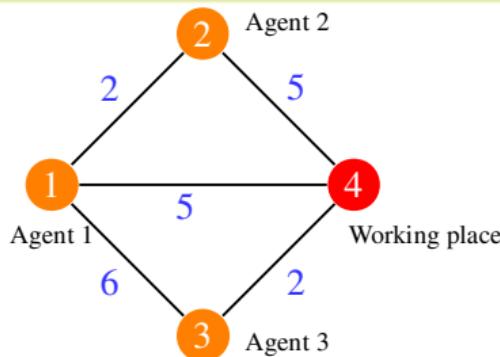
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$$2 = c(1, 2) - c(2) \leq c(1) - c(\emptyset) = 5,$$

and so on.

This game is concave!

$$(N, c) \in \mathcal{G}$$

Some interesting properties:

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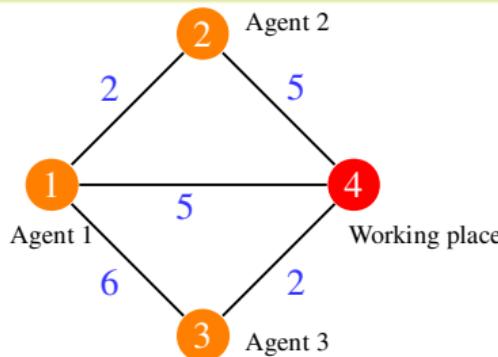
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Example 3.

$$N = \{1, 2, 3\}, \quad c(1) = 5, c(2) = 5, c(3) = 2, c(1, 2) = 7, c(1, 3) = 3, c(2, 3) = 7, c(1, 2, 3) = 9.$$

$$1 = c(1, 3) - c(3) < c(1, 2, 3) - c(2, 3) = 2$$

$(N, c) \in \mathcal{G}$

Some interesting properties:

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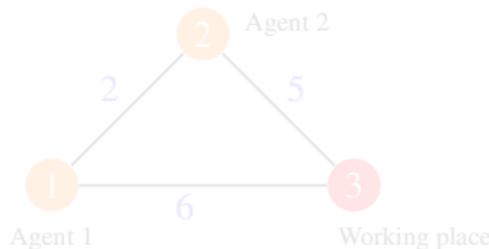
$$1 = \textcolor{red}{c}(1, 3) - \textcolor{red}{c}(3) < \textcolor{red}{c}(1, 2, 3) - \textcolor{red}{c}(2, 3) = 2$$

- A solution concept:

$$f : (N, c) \in \mathcal{G} \longrightarrow f(N, c) \subset \mathbb{R}^n.$$

$$(N, c) \in \mathcal{G} \longrightarrow I(N, c)$$

Example 1 (cont.).



$$N = \{1, 2\},$$

$$c(1) = 6, c(2) = 5, c(1, 2) = 7.$$

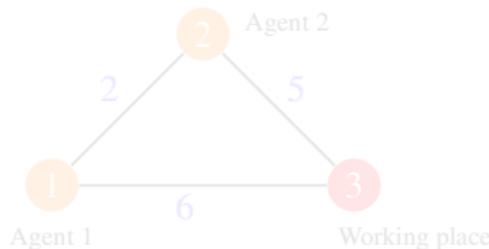
$$f(N, c) = I(N, c) = \text{conv}\{(6, 1), (2, 5)\}.$$
$$(,) \in (,)$$

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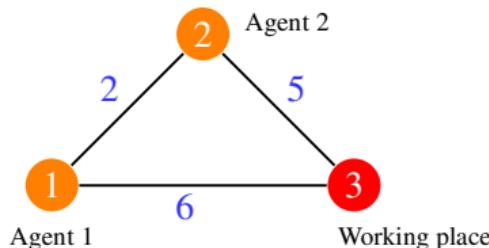
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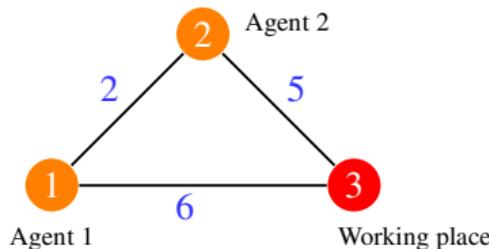
$(1, 3) \in \text{conv}\{(6, 1), (2, 5)\}$

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$$(N, c) \in \mathcal{G} \longrightarrow I(N, c)$$

Example 1 (cont).



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Example 4.

$$N = \{1, 2\}, \quad c(1) = 1, \quad c(\cdot) = 5, \quad c(1, 2) = 7.$$
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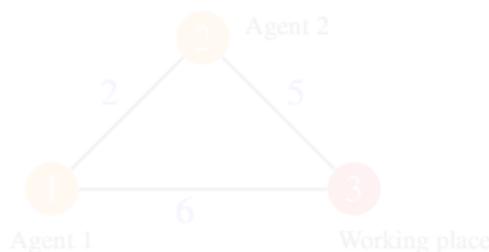
- A solution concept:

$$f : (N, c) \in \mathcal{G} \longrightarrow f(N, c) \subset \mathbb{R}^n.$$

If $f(N, c)$ always contains a unique element, f is called a **value**.
Otherwise, f is a **set-solution** concept.

A value f is **efficient** if and only if $\sum_{e \in N} f_e(N, c) = c(N)$, for all (N, c) .

Example 1 (cont.)



$$N = \{1, 2\},$$

$$c(1) = 6, c(2) = 5, c(1, 2) = 2.$$

$$f(N, c) = (v,) = \text{con } \{(1, 6), (2, 5)\}$$

$$(v,) = \{(1, 6), (2, 5)\}$$

$$(v,) = (6, 5)$$

Some solution concepts

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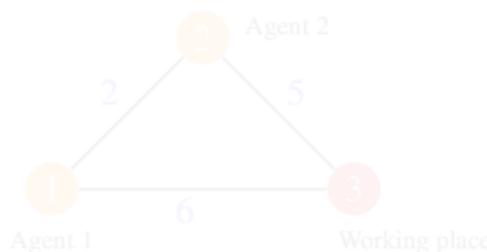
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$$f(N, c) = (v,) = \text{con } \{(1, 1), (2, 5)\}$$

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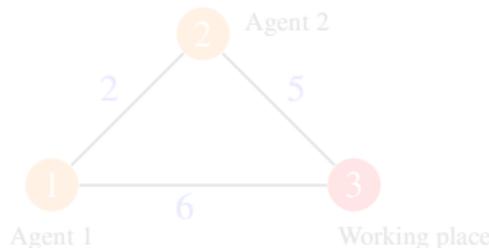
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Example 1 (cont).



$$N = \{1, 2\},$$

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$$f^1(N, c) = \{(N, c) = \text{com}\{(1, 1), (2, 5)\},$$

$$(,) = \{(,), (,)\}$$

$$(,) = (,)$$

Some solution concepts

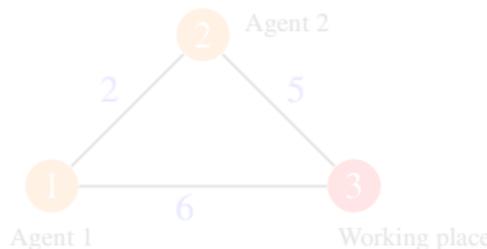
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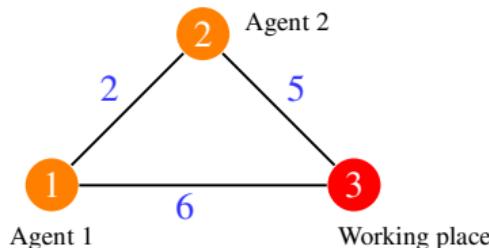
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$$f^2(N, c) = \text{com}\{(3, 4), (4, 3)\}.$$

$$f^3(N, c) = (4, 3)$$

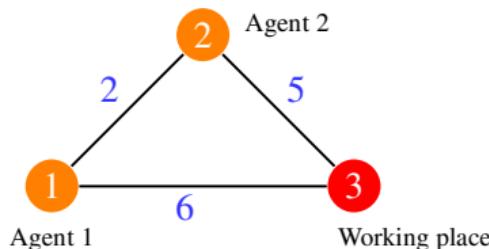
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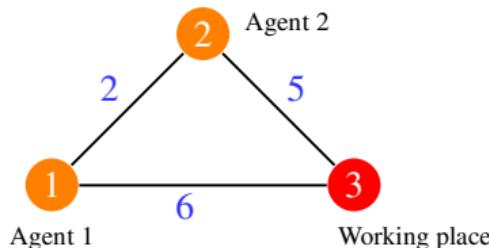
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A value f is **efficient** if and only if $\sum_{i \in N} f_i(N, c) = c(N)$, for all (N, c) .

Example 1 (cont).



$$\begin{aligned}N &= \{1, 2\}, \\c(1) &= 6, \quad c(2) = 5, \quad c(1, 2) = 7, \\f^1(N, c) &= I(N, c) = \text{conv}\{(6, 1), (2, 5)\}, \\f^2(N, c) &= \text{conv}\{(3, 4), (4, 3)\}, \\f^3(N, c) &= (4, 3)\end{aligned}$$