Game theory and OR games

Silvia Miquel and Cori Vilella

EURO Summer Institute
Lleida, July 28th 2014

Outline

1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

5 References

Decision problems

The study of decision problems posed in mathematical language.

OR

- One decision maker faces an optimization problem.
- *Linear Programming and Extensions*
  Dantzig (1963)
- Applications of the OR techniques:
  - Militar planning
  - Industry
  - Engineering
  - Economics

GT

- Situations involving at least two interacting decision makers (or players).
- *Theory of Games and Economic Behavior*
  Von Neumann and Morgenstern (1944)
- Conflict situations or cooperation:
  - **Noncooperative** game theory.
    No binding agreements.
  - **Cooperative** game theory.
    Enforceable binding agreements are possible.

Game theory

Noncooperative GT

Description

Specifying the options, incentives and information of the players.
Attemp to determine how they will play.
Each player chooses a strategy. The player's goal is to maximize her own payoff.

Main questions

- Are there equilibrium points?

Cooperative GT

Description

Specifying what payoffs each coalition can obtain by the cooperation of its members.There is communication between the players in order to make agreements to form coalitions.

Main questions

- How to execute the project in an optimal way (which coalition form)?
- How to allocate the total revenue among the agents?

Interaction

Interaction

Operations Reseach & Cooperative Game Theory

$\Downarrow$

Operations Research Games

Fixed tree games - Spanning tree games - Chinese potsman games - Travelling salesman games - Permutation games - Assignment games - Transportation games - Sequencing games - Linear production games - Flow games - ...

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

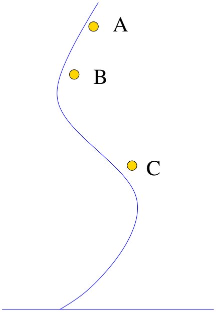
5 References

Outline

1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

Motivation (Rafels et al., 1999)

Three towns A, B and C are located next to the river.
They need to build a wastewater treatment plant for supplying drinking water to their populations.
Each town can build this plant on its own or along with other towns and then share the costs. Town B demand is half the one by A and C.
Suppose the river water flows naturally from A to B and C.



The costs for each situation are:

$$c(A) = 5.000$$
$$c(B) = 3.000$$
$$c(C) = 5.000$$

$$c(AB) = 5.100 + 900 = 6.000$$
$$c(AC) = 6.500 + 3.500 = 10.000$$
$$c(BC) = 5.100 + 1.900 = 7.000$$

$$c(ABC) = 6.500 + 2.100 + 1.900 = 10.500$$

Solutions

It seems worthwhile to cooperate and build just one plant.
But, how may they divide the total cost?

i) Based on the use of the plant and connections

$\alpha_A = \frac{2}{5}6.5 = 2.6$

$\alpha_B = \frac{1}{5}6.5 + \frac{1}{3}2.1 = 2$

$\alpha_C = \frac{2}{5}6.5 + \frac{2}{3}2.1 + 1.9 = 5.9$

Based on the use

$\alpha = (2.6, 2, 5.9)$

ii) Proportional to their cost with respect to the global cost

$\beta_A = \frac{5}{13}10.5$

$\beta_B = \frac{3}{13}10.5$

$\beta_C = \frac{5}{13}10.5$

Proportional to the cost

$\beta = (4.04, 2.42, 4.05)$

Motivation example. Solutions

iii) Based on the marginal cost

$$\gamma_A = \frac{10.5-7}{8.5}10.5$$

$$\gamma_B = \frac{10.5-10}{8.5}10.5$$

$$\gamma_C = \frac{10.5-6}{8.5}10.5$$

Based on the marginal cost

$\gamma = (4.35, 0.62, 5.56)$

iv) Proportional to the number of participants.They share the cost equally.

$$\delta_A = \delta_B = \delta_C = \frac{10.5}{3}$$

Proportional

$\delta = (3.5, 3.5, 3.5)$

v) Proportional to the water demand. We recall that the water demand follows the proportion $(2, 1, 2)$.

$$\epsilon_A = \epsilon_C = \frac{2}{5}10.5$$

$$\epsilon_B = \frac{1}{5}10.5$$

Proportional to the demand

$\epsilon = (4.2, 2.01, 4, 2)$

Looking for a nice solution

All those solutions are unacceptable from the cooperation point of view.

Let's define the game in a formal way and let's look for a nice solution for all players.

Cost game $\longrightarrow$ **Savings game**

$$\begin{array}{rccc} c: & 2^N & \to & \mathbb{R} \\ & S & \mapsto & c(S) \end{array}$$

$$\begin{array}{rccc} v: & 2^N & \to & \mathbb{R} \\ & S & \mapsto & v(S) = \sum_{i \in S} c(i) - c(S) \end{array}$$

$$N = \{A, B, C\}$$

$$N = \{A, B, C\}$$

$$\begin{array}{ccc} c(A) = 5 & c(B) = 3 & c(C) = 5 \\ c(AB) = 6 & c(AC) = 10 & c(BC) = 7 \\ & c(ABC) = 10.5 & \end{array}$$

$$\begin{array}{ccc} v(A) = 0 & v(B) = 0 & v(C) = 0 \\ v(AB) = 2 & v(AC) = 0 & v(BC) = 1 \\ & v(ABC) = 2.5 & \end{array}$$

Outline

1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

Transferable utility coalitional game

Definition

A *coalitional game* is an ordered pair (N, v) where

- $N = \{1, 2, ..., n\}$ is the set of players, and
- $v : 2^N \to \mathbb{R}$ is a real-valued function on the set 2^N of all subsets of N, named coalitions, with $v(\emptyset) = 0$.

The function v is called the characteristic function of the game and provides the worth $v(S)$ of each coalition $S \subseteq N$.

Such a game, (N, v), is called a cooperative game in characteristic function form.

Outline

1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

Looking for a nice solution. The imputation set

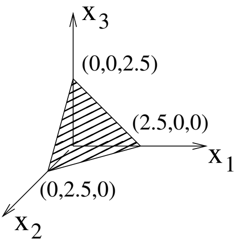
Definition

The *set of imputations* of a game (N, v) is the set

$$I(v) := \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

In our example,

$$\begin{aligned} I(v) := \{x \in \mathbb{R}^3 \text{ such that} \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and} \\ & x_1 + x_2 + x_3 = 2.5\} \end{aligned}$$



Imputation set. Example

Imputation set

$I(v) := \{x \in \mathbb{R}^3 \text{ such that } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and } x_1 + x_2 + x_3 = 2.5\}$

Do our solutions, α, β, ..., belong to the imputation set?

- $\alpha = (2.6, 2, 5.9) \quad \rightarrow \quad x_\alpha = (5 - 2.6, 3 - 2, 5 - 5.9) = (2.4, 1, -0.9) \notin I(v)$
- $\beta = (4.04, 2.42, 4.05) \quad \rightarrow \quad x_\beta = (0.96, 0.58, 0.95) \in I(v)$
- $\gamma = (4.35, 0.65, 5.56) \quad \rightarrow \quad x_\gamma = (0.65, 2.35, -0.56) \notin I(v)$
- $\delta = (3.5, 3.5, 3.5) \quad \rightarrow \quad x_\delta = (1.5, -0.5, 1.5) \notin I(v)$
- $\epsilon = (4.2, 2.01, 4.2) \quad \rightarrow \quad x_\epsilon = (0.8, 0.99, 0.8) \in I(v)$

We can easily check that all these solutions are efficient, but some of them don't satisfy the individual rationality.
What about those in the imputation set? Are they nice enough?

Outline

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

5 References

Looking for a nice solution. The core

Definition

The *core* of a game (N, v) is the set

$$C(v) := \{x \in \mathbb{R}^N | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}.$$

Notice that if a core allocation x is proposed, then no coalition S has an incentive to split off from the grand coalition N.

Definition

A game (N, v) is said to be *balanced* if it has a nonempty core.

Definition

A game (N, v) is said to be *totally balanced* if the core of every subgame is nonempty, where the subgame corresponding to some coalition $T \subseteq N$, $T \neq \emptyset$, is the game $(T, v_{|T})$ with $v_{|T}(S) = v(S)$ for all $S \subseteq T$.

The core. Example

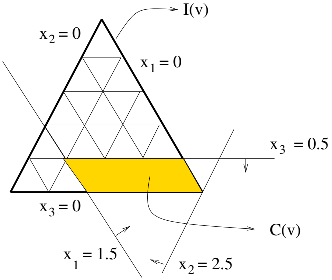
The game

$$N = \{1, 2, 3\}$$

$$v(1) = 0 \quad v(2) = 0 \quad v(3) = 0$$
$$v(12) = 2 \quad v(13) = 0 \quad v(23) = 1$$
$$v(123) = 2.5$$

The core

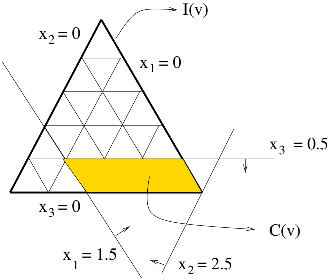
$$C(v) = \{x \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 2.5, \\ x_1 + x_2 \geq 2,\ x_1 + x_3 \geq 0,\ x_2 + x_3 \geq 1, \\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$$



The core. Example

The core

$$C(v) = \{x \in \mathbb{R}^3 \text{ such that}$$
$$x_1 + x_2 + x_3 = 2.5,$$
$$x_1 + x_2 \geq 2,\ x_1 + x_3 \geq 0,\ x_2 + x_3 \geq 1,$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$$



Do our solutions in the imputation set also belong to the core?

- $x_\beta = (0.96, 0.58, 0.95) \notin C(v)$ since $x_1 + x_2 \ngeq 2$.
- $x_\epsilon = (0.8, 0.99, 0.8) \notin C(v)$ since $x_1 + x_2 \ngeq 2$.

Core examples

Although the core of a game seems to be a good set-solution, it has a problem: it may be empty. In fact, the core may take different shapes... Which is the core of the following games?

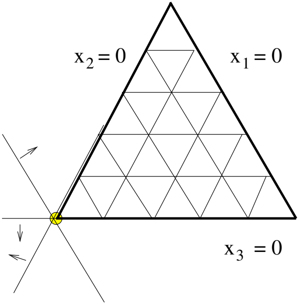
$v_A(i)=0$ $v_A(12)=5$ $v_A(13)=5$ $v_A(23)=0$ $v_A(123)=5$	$v_B(i)=0$ $v_B(12)=3$ $v_B(13)=2$ $v_B(23)=0$ $v_B(123)=5$	$v_C(i)=0$ $v_C(12)=3$ $v_C(13)=3$ $v_C(23)=0$ $v_C(123)=5$	$v_D(i)=0$ $v_D(12)=0$ $v_D(13)=0$ $v_D(23)=0$ $v_D(123)=5$
$v_E(i)=0$ $v_E(12)=3$ $v_E(13)=3$ $v_E(23)=4$ $v_E(123)=5$	$v_F(i)=0$ $v_F(12)=3$ $v_F(13)=2$ $v_F(23)=2$ $v_F(123)=5$	$v_G(i)=0$ $v_G(12)=3$ $v_G(13)=3$ $v_G(23)=3$ $v_G(123)=5$	$v_H(i)=0$ $v_H(12)=4$ $v_H(13)=4$ $v_H(23)=4$ $v_H(123)=5$

Example A

Let (N, v_A) be the 3-person game

$$v_A(i) = 0 \quad \forall i \in \{1, 2, 3\}$$
$$v_A(12) = 5 \quad v_A(13) = 5 \quad v_A(23) = 0$$
$$v_A(123) = 5$$

Then, $C(v_A)$ is one point $(5, 0, 0)$:



Example B

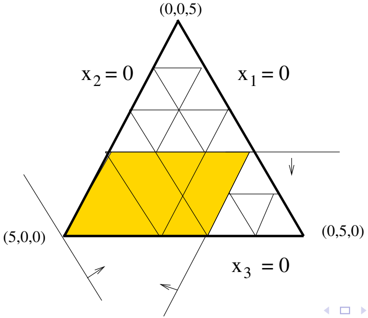
Let (N, v_B) be the 3-person game

$$v_B(i) = 0 \quad \forall i \in \{1,2,3\}$$

$$v_B(12) = 3 \quad v_B(13) = 2 \quad v_B(23) = 0$$

$$v_B(123) = 5$$

Then, $C(v_B)$ is the set:



Example C

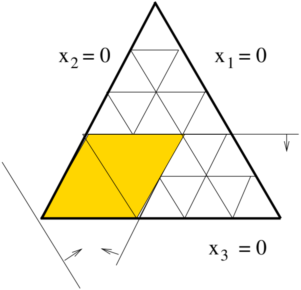
Let (N, v_C) be the 3-person game

$$v_C(i) = 0 \quad \forall i \in \{1, 2, 3\}$$

$$v_C(12) = 3 \quad v_C(13) = 3 \quad v_C(23) = 0$$

$$v_C(123) = 5$$

Then, $C(v_C)$ is the set:



Example D

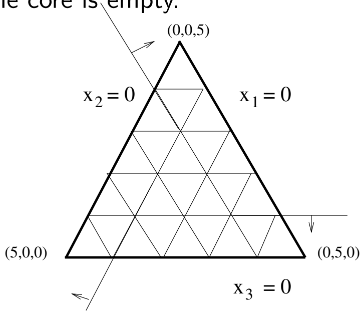
Let (N, v_D) be the 3-person game

$$v_D(i) = 0 \quad \forall i \in \{1, 2, 3\}$$

$$v_D(12) = 4 \quad v_D(13) = 4 \quad v_D(23) = 4$$

$$v_D(123) = 5$$

Then, $C(v_D) = \emptyset$, the core is empty.



Example E

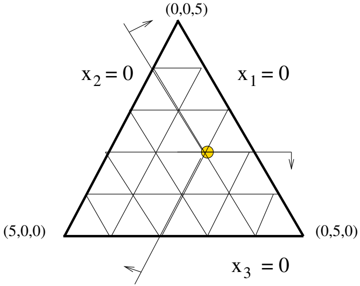
Let (N, v_E) be the 3-person game

$$v_E(i) = 0 \quad \forall i \in \{1,2,3\}$$

$$v_E(12) = 3 \quad v_E(13) = 3 \quad v_E(23) = 4$$

$$v_E(123) = 5$$

Then, $C(v_E)$ is one point $(1,2,2)$:



Example F

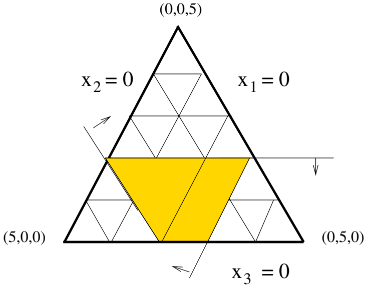
Let (N, v_F) be the 3-person game

$$v_F(i) = 0 \quad \forall i \in \{1,2,3\}$$

$$v_F(12) = 3 \quad v_F(13) = 2 \quad v_F(23) = 2$$

$$v_F(123) = 5$$

Then, $C(v_F)$ is the set:



Example G

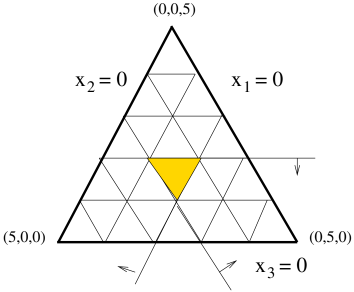
Let (N, v_G) be the 3-person game

$$v_G(i) = 0 \quad \forall i \in \{1, 2, 3\}$$

$$v_G(12) = 3 \quad v_G(13) = 3 \quad v_G(23) = 3$$

$$v_G(123) = 5$$

Then, $C(v_G)$ is the set:

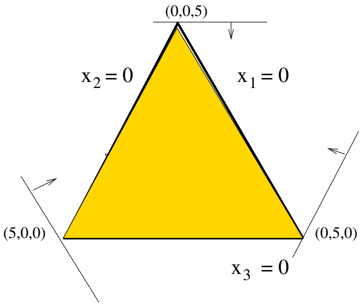


Example H

Let (N, v_H) be the 3-person game

$$v_H(i) = 0 \quad \forall i \in \{1, 2, 3\}$$
$$v_H(12) = 0 \quad v_H(13) = 0 \quad v_H(23) = 0$$
$$v_H(123) = 5$$

Then, $C(v_H)$ coincides with the imputation set:



Outline

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

5 References

Point solution concept

Both the set of imputations and the core of the game are solutions that may contain infinite points. This is the case of other solution concepts: the stable set, the bargaining set, the kernel, etc.

However, we may be interested on finding just one point, a point solution.

Let's now consider some point-solution concepts:

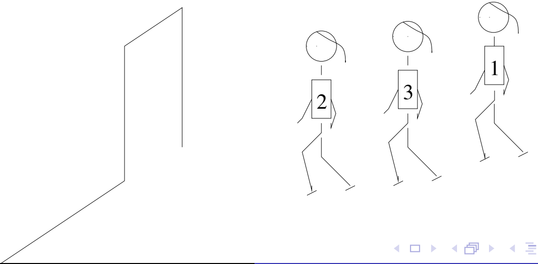
- The Shapley value $\phi(v)$.
  It is not always in the core.
- The nucleolus $\nu(v)$.
  It always belongs to the core.
- The τ-value.
  It may lie outside the core.

The Shapley value. Marginal contribution

Consider the folloging game (N, v) with $N = \{1, 2, 3\}$ and v such that:

$$v(1) = 6 \quad v(2) = 12 \quad v(3) = 18$$
$$v(12) = 30 \quad v(13) = 60 \quad v(23) = 90$$
$$v(\emptyset) = 0 \quad v(123) = 120$$

Let's imagine a procedure in which three players enter a room in any order. Each player that enters the room receives her marginal contribution to the coalition of players waiting for her in the room.



The Shapley value. Marginal contribution

Recall $v(1) = 6 \quad v(2) = 12 \quad v(3) = 18 \quad v(\emptyset) = 0$

$$v(12) = 30 \quad v(13) = 60 \quad v(23) = 90 \quad v(123) = 120$$

Suppose the 3 players enter the room in the order $\theta = (2, 3, 1)$. Then,

- The first player, player 2, gets

$$x_2 = v(2) - v(\emptyset) = 12.$$

- The second player, player 3, gets

$$x_3 = v(2, 3) - v(2) = 90 - 12 = 78.$$

- The third player, player 1, gets

$$x_1 = v(1, 2, 3) - v(2, 3) = 120 - 90 = 30.$$

So, when the order is $\theta = (2, 3, 1)$, the marginal contributions vector is

$$m^\theta = (x_1, x_2, x_3) = (30, 12, 78).$$

The Shapley value

Recall

$$v(1) = 6 \quad v(2) = 12 \quad v(3) = 18 \quad v(\emptyset) = 0$$

$$v(12) = 30 \quad v(13) = 60 \quad v(23) = 90 \quad v(123) = 120$$

Since there are three players in the game, there are $3! = 6$ possible orders.

Order of entry into the room θ	m^θ 1	2	3
1, 2, 3	6	24	90
1, 3, 2	6	60	54
2, 1, 3	18	12	90
2, 3, 1	30	12	78
3, 1, 2	42	60	18
3, 2, 1	30	72	18
	132	240	348

The Shapley value is $\phi(v) = (\frac{132}{6}, \frac{240}{6}, \frac{348}{6}) = (22, 40, 58)$

The Shapley value

Let, (N, v) be a coalitional game. Then its Shapley value is

$$\phi(v) = \frac{1}{n!} \sum_{\theta} m^{\theta}.$$

Our game

$$N = \{1, 2, 3\}$$

$$\begin{array}{cccc} v(\emptyset) = 0 & v(1) = 0 & v(2) = 0 & v(3) = 0 \\ v(123) = 2.5 & v(12) = 2 & v(13) = 0 & v(23) = 1 \end{array}$$

We wonder...

- Which is the Shapley value?
- Does it lie in the core?

The Shapley value. Axioms

The Shapley value $\phi(v)$ is the only solution that satisfies:

- **Efficiency** $\sum_{i \in N} \phi_i(v) = v(N)$.
- **Symmetry** If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j, then $\phi_i(v) = \phi_j(v)$.
- **Dummy axiom** If i is such that $v(S) = v(S \cup \{i\})$ for every coalition S not containing i, then $\phi_i(v) = 0$.
- **Additivity** If u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.

The nucleolus

Definition

As a mesure of the inequity of an imputation x for a coalition S is defined as the *excess (or complain)*,

$$e(x,S) = v(S) - \sum_{j\in S} x_j,$$

which mesures the amount by which coalition S falls short of its potential $v(S)$ in the allocation x.

Note that if $x \in C(v)$, then $\sum_{i\in S} x_i \geq v(S)$ for all S. Therefore, if $x \in C(v)$, all its excesses are negative or zero.

The nucleolus

Given an imputation x, for each $S \in 2^n$ except for $\emptyset$ and for N, we may compute the excess $e(x,S) = v(S) - x(S)$.

Our game

We take the Shapley value as the imputation x. So $x = \phi(v) = (\frac{5}{6}, \frac{4}{3}, \frac{1}{3})$.
Then the excesses are:

S	$v(S)$	$e(x,S)$
1	0	$-5/6$
2	0	$-4/3$
3	0	$-1/3$
12	2	$-1/6$
13	1	$-1/6$
23	0	$-5/3$

Further, these excesses are ordered in a decreasing order and we obtain the vector of excesses $\mathcal{O}(x) = (-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{3}, -\frac{5}{3}, -\frac{5}{6}, -\frac{4}{3})$.

The nucleolus

The vector $\mathcal{O}(x) \in \mathbb{R}^{2^n-2}$ is the vector of excesses arranged in decreasing order.
On these vectors we use the lexographic order.

Definition

We say a vector $y = (y_1, y_2, \ldots, y_k)$ is lexographically less than a vector $z = (z_1, z_2, \ldots, z_k)$, and write $y <_L z$, if

- $y_1 < z_1$, or
- $y_1 = z_1$ and $y_2 < z_2$, or
- $y_1 = z_1, y_2 = z_2$ and $y_3 < z_3$, or
- ..., or
- $y_1 = z_1, y_2 = z_2, y_3 = z_3, \ldots$ and $y_k < z_k$.

That is, $y <_L z$, if the first component in which y and z differ, that component of y is less than the corresponding component of z.

The nucleolus

The **nucleolus** is an efficient allocation that minimizes $\mathcal{O}(x)$ in the lexographic ordering.

Definition

Let (N, v) be a coalitional game. The nucleolus $\nu(v)$ is the set of imputations such that $\mathcal{O}(\nu) <_L \mathcal{O}(x)$ for all $x \in I(v)$.

Schmeidler (1969) proved that, if the game (N, v) is essential (that is to say, $I(v) \neq \emptyset$), the nucleolus always exists and it is unique.

The Nucleolus. Properties

- It always selects a core imputation if the core is non- empty. Preserves the cooperation.
- Selects the imputation that gives rise to smaller complaints by coalitions.
- If the core is only one point then this is the Nucleolus.
- **Symmetry** If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every coalition S not containing i and j, then $\nu_i(v) = \nu_j(v)$.
- **Dummy** If i is a dummy player, then $\nu_i(v) = v(i)$.

Outline

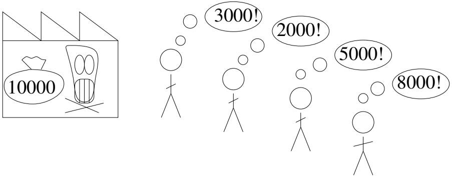
1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

Bankruptcy problem

The bankruptcy situation

Agents have claims on a resource that add up to more than what is available. How should the resource be divided?

An important application is to bankruptcy: a firm goes bankrupt and its liquidation value has to be allocated among its creditors.



Now they all demand their claim back, but there is not enough to satisfy all the claims.

Bankruptcy problem

Consider a situation where an estate E has to be divided among n claimants.

- The set of claimants is denoted by $N = \{1, 2, \dots n\}$.
- Claimant i advances a claim d_i on E.
  We'll consider $d_1 \leq d_2 \leq \cdots \leq d_n$ without loss of generality.

The problem is how to divide E among the claimants.

Definition

A *bankruptcy problem* is an ordered pair $(E; d) \in \mathbb{R} \times \mathbb{R}^n$, where

- $0 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ and
- $0 \leq E \leq d_1 + d_2 + \cdots + d_n =: D$

Bankruptcy game

Definition

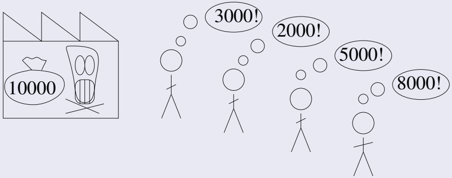
A cooperative game (N, v) is a *bankruptcy game* if there exists a bankruptcy problem $(E; d)$ such that

$$v(S) = \max\left\{0, E - \sum_{i \in N \setminus S} d_i\right\} \quad \text{for all } S \subseteq N$$

Bankruptcy game. Example

Example

A firm has gone bankrupt and leaves an amount of $E = 10000$ euro. There are only four lenders with the following claims on E:
$d_1 = 3000, d_2 = 2000, d_3 = 5000, d_4 = 8000$ euros.



Each coalition $S \subseteq N$ may take the amount after paying the claims to the claimants which are not in the coalition.
For instance, if $S = \{1, 4\}$, then

$$v(14) = \max\{0, E - d_2 - d_3\} = \max\{0, 10000 - 2000 - 5000\} = 3000$$

Bankruptcy game. Example

Example

A firm has gone bankrupt and leaves an amount of $E = 10000$ euro. There are only four lenders with the following claims on E:
$d_1 = 3000, d_2 = 2000, d_3 = 5000, d_4 = 8000$ euros.

Which is the characteristic function?

Bankruptcy game. Example

Example

A firm has gone bankrupt and leaves an amount of $E = 10000$ euro. There are only four lenders with the following claims on E:
$d_1 = 3000, d_2 = 2000, d_3 = 5000, d_4 = 8000$ euros.

Which is the characteristic function?

$$
\begin{array}{lll}
v(1) = 0, & v(12) = 0, & v(123) = 2000, \\
v(2) = 0, & v(13) = 0, & v(124) = 5000, \\
v(3) = 0, & v(14) = 3000, & v(134) = 8000, \\
v(4) = 0, & v(23) = 0, & v(234) = 7000, \\
 & v(24) = 2000, & \\
 & v(34) = 5000, & v(N) = 10000
\end{array}
$$

Monotonicity. Superadditivity. Convexity

Definition

- A cooperative game (N, v) is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T$.
- A cooperative game (N, v) is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.
- A cooperative game (N, v) is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \in 2^N$.

Bankruptcy games are monotonic, superadditive and convex games.

Theorem

If (N, v) is **convex**, then the Shapley value belongs to the core,

$$\phi(v) \in C(v).$$

Bankruptcy game. Shapley value

Example

A) Let's consider a bankruptcy game where the estate is $E = 300$ and the claims are $d_1 = 100, d_2 = 200$ and $d_3 = 300$.

Example

B) Let's consider a bankruptcy game where the estate is $E = 300$ and the claims are $d_1 = 200$ and $d_2 = 300$.

Which is the characteristic function for example A and B?
Can you find their Shapley value?
Does the Shapley value belong to the core?

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

5 References

Outline

1 OR and GT

2 Cooperative game theory
- Motivation
- Coalitional games
- The imputation set
- The core
- Point solutions
- Bankruptcy games

3 Operations Research Games
- Tree games

4 Game theory group

5 References

Minimum cost spanning tree games

Claus and Kleitman (1973) studied the following situation:

- Several customers who are geographically separated have to be linked to a certain supplier.
- The supplier could be for instance an electricity plant or a well, and the customers could be various towns.
- A customer can be linked directly to the supplier or through other customers. Each link induces a non-negative cost.

The question

How to allocate the total cost, incurred by connecting all the customers to the supplier, among the customers?

Minimum cost spanning tree games

Formally, the situation is described as follows.

- $N = \{1, 2, \dots, n\}$ is the set of customers and the supplier is denoted by 0.
- $G_N = (N \cup \{0\}, E_N)$ is the complete graph with set of nodes $N \cup \{0\}$ and set of edges E_N.
- To each edge $e_{ij} = e_{ji} \in E_N$ with end points $i, j \in N \cup \{0\}$, a cost c_{ij} is attached. Where c_{ij} is the cost induced when i and j are linked.

The problem of finding a cheapest way to connect all the customers to the supplier is equivalent to the problem of finding a *minimum cost spanning tree* of the graph.
The cost of a coalition $S \in 2^N$ is defined to be the cost of linking all members of S to the supplier in a cheapest possible way without using links involving customers not in S (Bird, 1976).

Minimum cost spanning tree games

Definition

A cooperative game c is a **minimum cost spanning tree game** if there exists a graph $G_N = (N \cup \{0\}, E_N)$ with costs $c_{ij} \geq 0$ attached to each edge e_{ij}, where $i, j \in N \cup \{0\}$.

Such that for each $S \in 2^N$, $c(S)$ is the cost of the minimum cost spanning tree of the graph G_S.

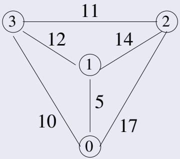
Where G_S is obtained from G_N removing all nodes in $N \backslash S$ and all edges with at least one endpoint in $N \backslash S$.

Minimum cost spanning tree game. Example

Example

Let's consider three players $N = \{1, 2, 3\}$, who want to be connected to the well, 0. In the following graph,

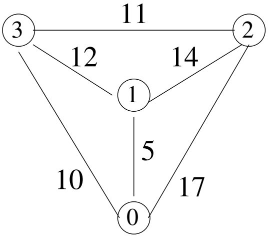
- nodes are occupied by different players and the well is the node 0,
- the cost of every connection is next to the corresponding edge.



Which is the minimum cost spanning tree game corresponding to this situation?

Minimum cost spanning tree games

Which is the minimum cost spanning tree game corresponding to this situation?

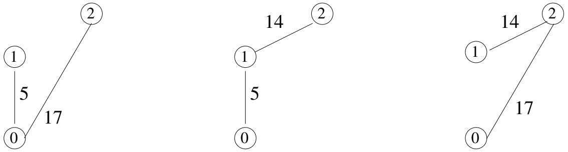


Clearly, $c(i) = c_{i0}$. That is to say,

$$c(1) = 5 \qquad c(2) = 17 \qquad c(3) = 10$$

Minimum cost spanning tree games

How we find $c(12)$, $c(13)$, $c(23)$?

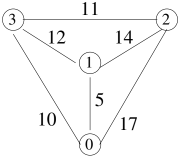


So, $c(12) = \min\{5+17, 5+14, 14+17\} = 19$.

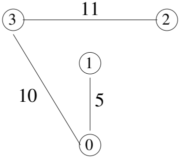
Similarly, we find the cost for the other two-player coalitions.
$c(13) = \min\{5+10, 5+12, 10+12\} = 15$ and
$c(23) = \min\{17+10, 17+11, 10+11\} = 21$

Minimum cost spanning tree games

Finally, what is the cost for the grand coalition, $c(123)$?



Now we should consider all the possible ways of connecting the three players to the well. Thus,
$c(123) = \min\{32, 33, 26, 34, 39, 29, 41, 31, 30, 28, 42, 43, 40, 33, 35, 36\}$.
So, $c(123) = 26$. Indeed, the minimum cost spanning tree for the grand coalition is



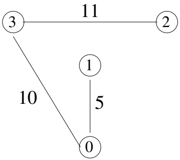
Minimum cost spanning tree games

Notice that, for each coalition S, c(S) is obtained from a graph with

- nodes: $S \cup 0$,
- edges: the necessary edges to connect the S nodes with the well.

This graph is always a tree (if there were a cycle, the cost wouldn't be minimum).

In the example above, if we consider the coalition $S = \{123\}$, the minimum cost spanning tree is



Minimum cost spanning tree games. The core

Once we have the characteristic functions, the main question is wether the core of the game is nonempty.
Since it is a cost game, the core allocations $(x_1, x_2, x_3) \in \mathbb{R}^3$ must meet the following restrictions:

The core

$$x_1 + x_2 + x_3 = 26$$
$$x_1 \leq 5$$
$$x_2 \leq 17$$
$$x_3 \leq 10$$
$$x_1 + x_2 \leq 19$$
$$x_1 + x_3 \leq 15$$
$$x_2 + x_3 \leq 21$$

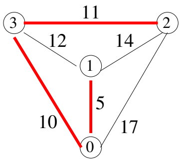
Does exist any core allocation?

Minimum cost spanning tree games. The core

Proposition (Bird, 1976)

Let (N, c) be a minimum cost spanning tree game. Let G_N be the correspondig graph. The Bird allocation, which consist of assigning to each player the connection cost to its predecessor in the tree with minimum cost for the grand coalition, always belongs to the core.

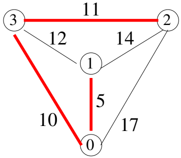
Recall our situation



- Which is the Bird allocation?
- Does it belong to the core?

Minimum cost spanning tree games. The core

Recall our situation



- Which is the Bird allocation? The Bird allocation is $(5, 11, 10)$.
- Does it belong to the core? We can check that $(5, 11, 10) \in C(c)$. Where,

$$C(c) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ such that} \begin{array}{r} x_1 \leq 5 \\ x_2 \leq 17 \\ x_3 \leq 10 \\ x_1 + x_2 \leq 19 \\ x_1 + x_3 \leq 15 \\ x_2 + x_3 \leq 21 \\ x_1 + x_2 + x_3 = 26 \end{array} \right\}$$

Minimum cost spanning tree games. Properties

Definition

A cooperative cost game (N, c) is **concave** if, for all player $i \in N$, it holds that $\forall S \subseteq T \subseteq N \backslash \{i\}$,

$$c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T).$$

Minimum spanning tree games:

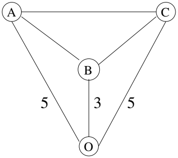
- The core is nonempty since the Bird allocation belongs to the core.
- They are not always concave games. For instance, in the example above, $c(12) - c(2) < c(123) - c(23)$.

The motivation example

Recall the motivation example:
$N = \{A, B, C\}$

$$\begin{array}{ccc} c(A) = 5 & c(B) = 3 & c(C) = 5 \\ c(AB) = 6 & c(AC) = 10 & c(BC) = 7 \\ & c(ABC) = 10.5 & \end{array}$$

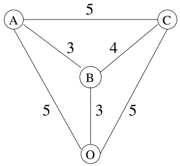
- Is it a minimum cost spanning tree game?
  If it was a mcst game, since $c(A) = 5, c(B) = 3$ and $c(C) = 5$, we would have the following underlying graph:



The motivation example

Motivation example:
$$c(A)=5 \qquad c(B)=3 \qquad c(C)=5$$
$$c(AB)=6 \qquad c(AC)=10 \qquad c(BC)=7$$
$$c(ABC)=10.5$$

- Is it a minimum cost spanning tree game?
  Now, since $c(AB)=6, c(AC)=10$ and $c(BC)=7$, the underlying graph would be:



But then, the cost for the grand coalition would be $c(ABC)=10$
So, it is not a minimum cost spanning tree game (mcst).

In order to be a mcst game, it should be
$$c(A)=5 \qquad c(B)=3 \qquad c(C)=5$$
$$c(AB)=6 \qquad c(AC)=10 \qquad c(BC)=7$$
$$c(ABC)=10$$

1. OR and GT
2. Cooperative game theory
   - Motivation
   - Coalitional games
   - The imputation set
   - The core
   - Point solutions
   - Bankruptcy games
3. Operations Research Games
   - Tree games
4. Game theory group
5. References

Grup de recerca

Grup de recerca de la UB a la Facultat d'Econòmiques i Empresarials:
Teoria de jocs, investigació operativa i optimització ▶

References

1. Mas-Colell A, Whinston MD and Green JR (1995)
Microeconomic theory
Oxford University Press
1. Bietenhader T and Okamoto Y (2006)
Core stability of minimum coloring games
Mathematics of Operations Research 31: 418-431
2. Deng X, Ibaraki T, Nagamochi H and Zang W (2000)
Totally balanced combinatorial optimization games
Mathematical Programming 87: 441-452
3. Okamoto Y (2003a)
Submodularity of some classes of the combinatorial optimization games
Mathematical Methods of Operations Research 58: 131-139
4. Okamoto Y (2003b)
Fair core allocations under conflicts -a game-theoretic point of view-
Proc. 14th ISAAC, Lec. Notes Comp.Sci. 2906: 686-695
5. Seinsche D (1974)

Thanks

Gràcies per la vostra atenció
Thank you for your attention