MAROTO ÁLVAREZ, C., [et al] (2014) Operations research in business administration and management. Valencia: Universitat Politècnica de València.

CHAPTER 7

DISCRETE MULTIPLE CRITERIA DECISION MAKING TECHNIQUES

Multiobjective programming and goal programming are applicable to both continuous and discrete problems. In other words, they can be used for making decisions when the number of alternatives is infinite but also when the number is finite and usually small. However, there are a number of methods specifically designed for this latter case that have been applied in many decision problems. In this chapter we will see the Analytical Hierarchy Process (AHP) and the *PROMETHEE* method.

Discrete multiple criteria decision problems can be classified into three groups. First, those in which we want to select only one of the alternatives. For example, which car or mobile phone to purchase, based on economic, technical and design criteria. Another group of problems consists of those in which we want to classify the alternatives, such as grouping the suppliers of a company as good, acceptable and bad, through various attributes (cost, technical, delivery time, etc.). Other problems are those in which our interest is focused on ordering alternatives by priority. For example, to prioritize investment projects for budget allocation.

7.1. ANALYTIC HIERARCHY PROCESS

7.1.1. INTRODUCTION

The Analytic Hierarchy Process, known as AHP, developed by Thomas L. Saaty (1980) has been successfully applied to a wide variety of decision making problems in companies and public administration. Amongst its applications we find strategic planning, resource allocation and selection, market share, production, business ethics, energy, health, education, environment and politics. It is also used to determine the weights in other techniques such as multiobjective and goal programming, as well as PROMETHEE and multi-attribute utility analysis.

As we have seen in the previous chapter, in a **multiple criteria context the optimal concept does not exist**. In general, we can say that the multiple criteria techniques are helpful in the decision making process, which seeks **to integrate the behaviour of the objectives with the judgment** of the decision maker/s, so as to be able to manage and make that subjectivity explicit.

In practice, many decision problems are not presented in a structured way with a list of objectives and alternatives, ready to make a systematic analysis. The choice of the criteria on which we will base our decision is subjective so we must make them explicit and the process transparent. This is particularly important for group or collaborative decision-making. An example would be when you have to select the best suppliers, taking into account the opinion of several people in the company. The AHP method considers the preferences of the decision maker/s through judgments about the relative importance of the criteria and the alternatives taken "in pairs". To apply this approach, quantitative

information on the outcome of each alternative on each of the criteria considered is not required, but only the value judgments of the decision maker/s.

7.1.2. BUILDING A HIERARCHICAL MODEL

The first phase of the AHP method consists of building the **decision hierarchy** that represents the multiple criteria decision problem. There are few rules for building hierarchies. The upper level consists of only one element, which is the overall aim of the problem. Successive levels may have multiple elements, all of the same order of magnitude, otherwise they must be at different levels. There is no limit to the number of levels in a hierarchy. If we cannot compare elements in relation to the next level above, one must determine search terms by which they can be compared and put them into an intermediate level. At this stage it is important to identify the actors involved in decision-making, and in particular its objectives and preferences. A widely used technique to design the hierarchy of a decision problem is brainstorming.

Let us take a simple example. A student of Business Administration and Management decides to study their final year degree in another European country through the Erasmus program. After reviewing the possible destinations he/she determines that his/her choices are Aarhus Universitet (Denmark), Universitet Gent (Belgium) and Radboud Universiteit Nijmegen (Holland).

Assuming that the cost of living in different countries is not considered, since the grant already takes that into account, the criteria to consider when making the decision are related to education and leisure. One of the student's objective is to improve his/her English. The prestige of the university is important for his/her Curriculum Vitae, as is the quality of teaching and the number of subjects available in English. Entertainment is also very important because relations with people from different cultures give a lifetime experience. The friendliness of the people, the possibility of going out and travelling through Europe are also relevant issues to the student.

Figure 7.1 shows the decision hierarchy of the student that intends to prioritize the Erasmus destinations that best suit his/her objectives. In general, the approach to building a hierarchy depends on the kind of decision to be made. If it comes to selecting or prioritizing alternatives, we can start from the lower level, choosing the alternatives first. Previous levels would include criteria for evaluating these alternatives and on the upper level there would be a single element, which is the overall goal. Sometimes the hierarchy is designed from top to bottom. In many real problems criteria and objectives that must inform the decision making are not known and the AHP method helps us identify them. There is no limit to the number of levels in a hierarchy. The question to be answered is: "Is it possible to compare the items that are placed on the same level in terms of any of the elements of the next higher level?". If the answer is no, we should decide in what terms they can be compared and create an intermediate level.
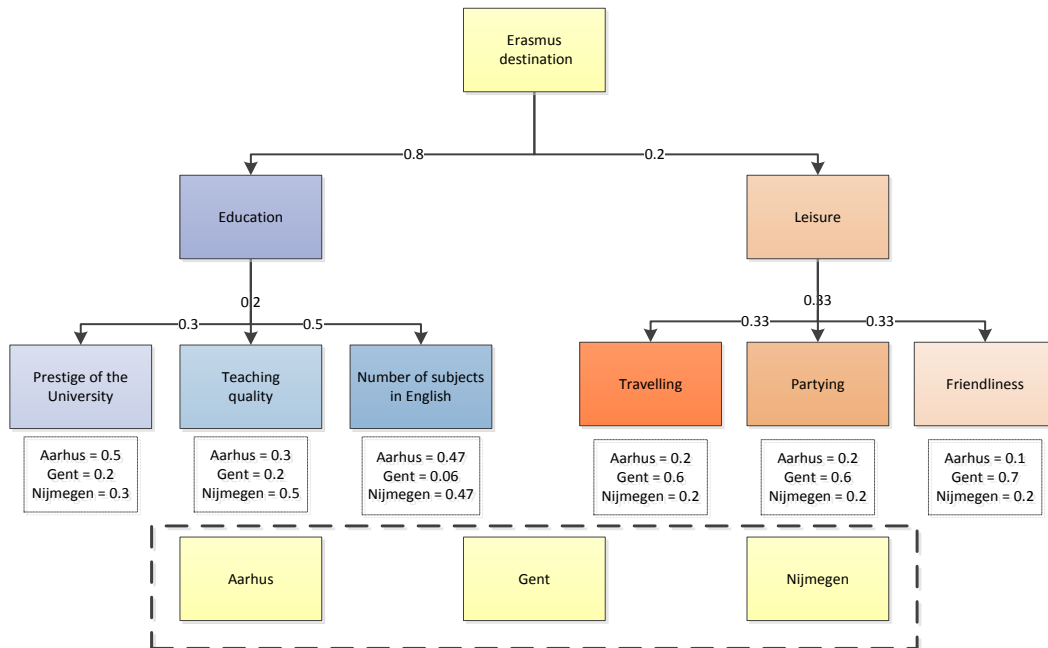
Figure 7.1. Decision hierarchy to prioritize the Erasmus destinations

The student must design the hierarchy that best represents the criteria to be taken into account and alternatives, i.e. all Erasmus destinations that he/she wants to evaluate. One should not consider more than nine destinations, and with not more than nine criteria in the same group in the hierarchy, as it has proven very difficult for people to make pairwise comparisons with such a large number of elements. Another rule to remember is that the elements of the same group must have the same order of magnitude.

7.1.3. SETTING PRIORITIES

In our example, the student's problem is to select in which university he/she should undertake his/her final year of studies. However, the number of places that are offered by the universities is small and other students may also apply for the same destination. Therefore, a prioritized list of the destinations he/she wishes to attend should be made.

The second phase of the AHP method consists of setting priorities between the elements of the hierarchy, then synthesizing our judgments to obtain global priorities to allow us to reach a final decision. For that, we perform pairwise comparisons of elements of the same level regarding the element of the next higher level. In our example, at the second level, the student would ask, how much more important is education compared to leisure when choosing an Erasmus destination? or, on the third level, how much more important is the prestige of the university with respect to the quality of teaching in education?

We can represent the results of pairwise comparisons in a matrix. The matrix is a simple method that enables us to collect information about the judgments and analyze their consistency. To complete the comparison matrix we will use numbers that represent the importance of one element over another. Table 7.1 presents the fundamental scale pairwise comparisons in the AHP method.

Table 7.1. Fundamental scale for pairwise comparisons in the AHP method

Intensity of importance or contribution of one activity over the other	Definition	Explanation
1	Equal importance	The two elements contribute equally to the objective
2		Intermediate importance between 1 and 3
3	Weak importance of one over another	Experience and judgment slightly favour one element over another
4		Intermediate importance between 3 and 5
5	Essential or strong importance	Experience and judgment strongly favour one element over another
6		Intermediate importance between 5 and 7
7	Demonstrated importance	An element is strongly favoured and its dominance is demonstrated in practice
8		Intermediate importance between 7 and 9
9	Absolute importance	The evidence favouring one element over another is of the highest possible order of affirmation
1/2 1/3 ... 1/8 1/9	If the **first element** has a **strong importance** when compared to the **second element,** we assign a **5 on the scale**. If we make the comparison of **the second element in relation to the first,** the value assigned on the scale is **1/5**

In our example to compare the three universities located in Aarhus, Gent and Nijmegen, with relation to the number of subjects available in English that we can take, we will fill a 3x3 matrix, whose values are shown in Table 7.2. The criterion "number of courses available in English" is written in the upper left corner and destinations are written in the first row and first column in the same order. This matrix has nine elements and all elements of the main diagonal are 1, since they represent the comparison of each university with itself. Of the remaining six elements, you have to complete only the three judgments above the main diagonal. The judgments below are their reciprocals.

Then students should ask themselves: How much greater is the number of subjects available in English at one university than in another? In Denmark all college degrees are taught in both English and Danish. In Holland college teaching also is performed in English. By contrast, in Gent, Flemish is the main language used in higher education, so

the availability of courses in English is lower. Note that the value assigned to an element ij in the matrix is the one resulting from the comparison of the element of row i with the column j. For example, we have assigned a 7 to indicate the comparison between the number of subjects in English from the University of Aarhus and Gent, as there are many more courses available in the first university. If the element of the row is less important than the value of the column its value is a fraction. For example, 1/7 when comparing Gent with Nijmegen.

Table 7.2. Comparison Matrix for Erasmus destinations by number of subjects in English

Number of subjects in English	Aarhus	Gent	Nijmegen
Aarhus	1	7	1
Gent	1/7	1	1/7
Nijmegen	1	7	1

In summary, the matrix of Table 7.2 and any pairwise comparison matrix satisfy the following properties:

1. $a_{ij} \geq 0 \; \forall \, i, j$

2. $a_{ij} = \frac{1}{a_{ji}}, \; \forall \, i, j$

3. $a_{ii} = 1 \; \forall \, i$

The AHP method requires that each matrix element a_{ij} takes one of the values of the fundamental scale comparisons 1/9, 1/8…1/2, 1, 2, 3… 9.

Once the pairwise comparison matrix has been obtained we can set the relative priorities of the Erasmus destinations on the number of courses available in English. The following method provides a rough estimate of the priorities that we will use in making the decision.

When calculating priorities, we first add the values in each column (Table 7.3). We then divide each cell in each column by the total of the column, in order to obtain a normalized matrix (Table 7.4).

Table 7.3. Calculation of the priorities of the Erasmus destinations by number of subjects in English

Number of subjects in English	Aarhus	Gent	Nijmegen
Aarhus	1	7	1
Gent	1/7	1	1/7
Nijmegen	1	7	1
Column Total	15/7	15	15/7

Table 7.4. Normalized matrix for Erasmus destinations by number of subjects in English

Number of subjects in English	Aarhus	Gent	Nijmegen
Aarhus	7/15	7/15	7/15
Gent	1/15	1/15	1/15
Nijmegen	7/15	7/15	7/15

Finally, we calculate the average of the rows by adding all of the values in each row of the normalized matrix and dividing the sum by the number of cells that are in a row, obtaining **local priorities** of Erasmus destinations for the criterion "number of subjects in English".

$$Aarhus = \frac{^7/_{15} + ^7/_{15} + ^7/_{15}}{3} = \frac{7}{15} = 0.47$$

$$Gent = \frac{^1/_{15} + ^1/_{15} + ^1/_{15}}{3} = \frac{1}{15} = 0.06$$

$$Nijmegen \frac{^7/_{15} + ^7/_{15} + ^7/_{15}}{3} = \frac{7}{15} = 0.47$$

The sum of local priorities should be equal to 1. We follow the same procedure to complete all matrices needed to solve the problem and calculate local priorities of the elements considered in our decision hierarchy. After calculating all local priorities of all pairwise comparison matrices in the hierarchy we can obtain the **global priorities**, the sum must also be equal to 1.

Assuming that local priorities for a student are the values shown in Figure 7.1, we calculate the global priorities of the alternatives weighting their local priorities with the priorities of the objectives. The results are as follows

$$Aarhus = 0.8 * [(0.3 * 0.5) + (0.2 * 0.3) + (0.5 * 0.47)] +$$
$$0.2 * [(0.33 * 0.2) + (0.33 * 0.2) + (0.33 * 0.1)] = 0.39$$

$$Gent = 0.8 * [(0.3 * 0.2) + (0.2 * 0.2) + (0.5 * 0.06)] +$$
$$0.2 * [(0.33 * 0.6) + (0.33 * 0.6) + (0.33 * 0.7)] = 0.23$$

$$Nijmegen = 0.8 * [(0.3 * 0.3) + (0.2 * 0.5) + (0.5 * 0.47)] +$$
$$0.2 * [(0.33 * 0.2) + (0.33 * 0.2) + (0.33 * 0.2)] = 0.38$$

Taking into account these global priorities the best Erasmus destination for the student of business administration and management in our example is Aarhus Universitet. The second option would be Nijmegen and lastly Gent.

There are two ways to synthesize the priorities, the **distributive mode** and the **ideal mode**. In the distributive mode the sum of the priorities of the alternatives equals one. It is used when there is dependence between the alternatives and the unit must be distributed between them. However, the ideal mode is the one that should be used when the objective is to select only one of the alternatives. In this case the local priorities of the alternatives are divided by the largest value and this is done for each criterion in such a way that for each criterion there is an alternative that is considered ideal. In both modes the priorities are weighted in the same way with the weights of the objectives. The difference between the two methods is more interesting in theory than in practice.

7.1.4. LOGICAL CONSISTENCY

In the decision making processes it is important to know the consistency of the judgments contained in the comparison matrices, since we do not want our decisions to be based on very inconsistent judgments. For example, it would not be logical that our student knows that the universities of Aarhus and Nijmegen have the same number of courses available in English, and, that when comparing these two universities with Gent, one university has moderate and the other strong importance.

When a matrix is consistent, the average of the sum of each row of the normalized matrix indicates how much the element of the row dominates the others in relative terms. On the other hand, the sum of the columns of the pairwise comparison matrix determines the degree to which each element is dominated by the other elements, so that the product of the two values is equal to one. When a matrix is consistent the elements of any column of the normalized matrix give the same priorities that we obtain by calculating the average

of the rows. This means that they are identical to the priorities of pairwise comparison matrix.

Multiplying the sum of the pairwise comparison matrix columns by the average of the sum of each row of the normalized matrix is 1, since they are reciprocal numbers. The sum of all these multiplications is equal to n, if the matrix is consistent. In our example the sum is equal to 3 which represent the three universities considered as alternatives for the destination of the Erasmus student.

As we have seen in the example, the pairwise comparison matrix, **A**, is formed through the comparison of each element to another. If we have n elements (criteria, objectives, alternatives), where their weights or priorities are $w_1,\ w_2\dots\ w_n$ the pairwise comparison matrix has the following structure:

$$A = \begin{pmatrix} 1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & 1 & \dots & w_2/w_n \\ \vdots & \vdots & \vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & 1 \end{pmatrix}$$

where $a_{ij} = {w_i}/{w_j}$. This is the relative weight of element i to element j.

The consistency of the judgments is related to the **transitivity of preferences** in the comparison matrix. In summary, a matrix **A** is **consistent** if

$$a_{ij} = a_{ik} * a_{kj}, \forall\ i, j, k\ .$$

Other notable properties are as follows:

1. The rank of matrix **A** is 1 as all the rows are multiples of the first row.

2. The eigenvalues of matrix **A** ($\{\lambda \in R: \det (A - \lambda I) = 0\}$) are all zero except for 1 as the rank of the matrix is 1.

3. The trace of matrix A is equal to n, as the diagonal consists of values that are all 1. Therefore, the only nonzero eigenvalue is n, since the sum of the eigenvalues of the matrix coincides with the trace.

4. The eigenvector associated to n coincides with the priority vector

    $\mathbf{w} = (w_1, w_2 \ldots w_n)$. This is $\mathbf{w}$ the nontrivial solution of the system.

    $\mathbf{A}\,\mathbf{w} = \lambda\,\mathbf{w}$

    Only the priority vector whose values sum to 1 is considered.

5. If we use priorities which sum 1, any column of the normalized matrix A coincides with the priority vector.

Priorities are calculated from judgments of pairwise comparison, which are given on a numerical scale. The values of pairwise comparison matrix $\boldsymbol{a_{ij}}$ can be considered as an estimation of the true values of w_i/w_j. In this situation n cannot be an eigenvalue of the matrix and therefore we must find the largest eigenvalue λ_{max} of the obtained matrix.

In summary, a pairwise comparison matrix $\mathbf{A}$ is consistent if and only if

$\lambda_{max} = n$.

$\lambda_{max} \geq n$ and there is a positive components vector $\mathbf{w^{max}}$.

There are several methods to obtain the weights vector $\mathbf{w^{max}}$ and λ_{max}, both exact and approximate. An exact method is calculated directly with the spreadsheet.

The approximate method that we used in the example is based on the property number 5. First we normalize each column of matrix $\mathbf{A}$, by adding all values in each column and dividing each cell by the total. Thus we obtain $\mathbf{A}_{norm}$. We then calculate the average of each row of $\mathbf{A}_{norm}$, with the sum of the values in each row and divide this by the number of rows, obtaining in this way the weight vector $\mathbf{w}$. We then calculate the product $\mathbf{Aw}$ and in the last step we calculate λ_{max}.

Inconsistencies can occur for two reasons: One because the decision maker establishes intransitive relationships in pairwise comparisons or because it changes the sense of preference. Although we cannot be so confident in our judgments as to force consistency in the pairwise comparison matrix, we need some degree of consistency in setting priorities for the elements with respect to some criterion to obtain valid results in real applications.

The eigenvalue is used to measure the degree of inconsistency, since, if the comparison matrix is consistent, the largest eigenvalue is equal to n. We define **Consistency Index (CI)** as follows:

$CI = (\lambda_{max} - n)/(n-1)$

This index represents the cumulative average of inconsistency of the matrix. To find out if it is large or small it is compared to the Random Consistency Index (RCI), which is the average value of CI of pairwise comparisons matrices of the same order randomly obtained. RCI values for the most common sizes of matrices are:

Table 7. 5. Random Consistency Index values for n size of matrices

n	2	3	4	5	6	7	8	9	10
RCI	0.00	0.58	0.90	1.12	1.25	1.35	1.41	1.45	1.51

Inconsistency Index (II) is defined as the ratio of the matrix CI and RCI,

II = CI/RCI.

The consistency of the comparison matrix is considered to be acceptable if the ratio is **less than or equal to 0.10**.

7.1.5. SOFTWARE

We can apply the AHP using a spreadsheet. However there are computer programs with graphics capabilities that allow us to enter, display the results and make sensitivity analysis, e.g. ***Expert Choice*** (expertchoice.com) and ***Super Decisions*** (http://www.superdecisions.com). In particular, the latter software allows us to solve decision problems using AHP and its generalization ANP (Analytic Network Process). ANP is used when the problem cannot be structured hierarchically because there are dependencies and interactions between its components, thus the problem can be best represented by a network than by a hierarchy.

In ***Expert Choice*** once we have defined the decision hierarchy, the software allows us to enter **data** for the comparison matrix **verbally**, **numerically** or **graphically** and it also generates the questionnaire. With *Expert Choice* one can see the necessary degree of detail, the local and global priorities and graphical reports for the sensitivity analysis. In other words, we can look at how the global priorities of the alternatives change when criteria weights change. In the following figures some of the possibilities which are offered by the program are represented using the Erasmus destination example.

In Figures 7.2 and 7.3 we can see the different ways of entering data for our example in *Expert Choice Comparion Suite*. We can introduce data graphically, numerically or verbally. The software generates the questionnaire for pairwise comparisons (Figure 7.2).
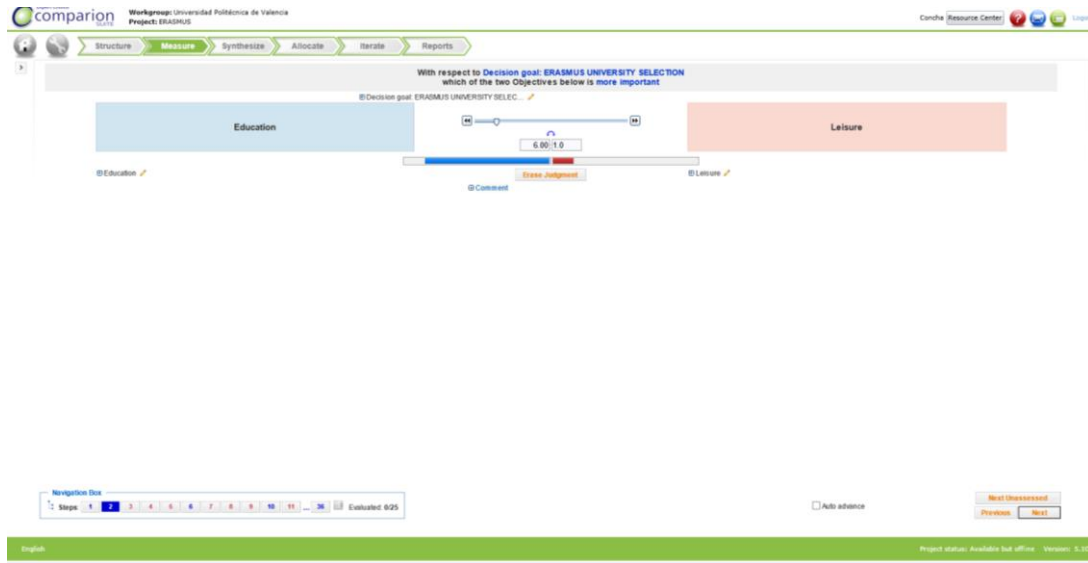
Figure 7.2. Entering data graphically/numerically in Expert Choice for Erasmus destination example

Figure 7.3. Entering data verbally in Expert Choice for Erasmus destination example

In Figure 7.4 we see the results and different types of graphs that can be used to perform sensitivity analysis. For example, by changing the bars that represent the weights that the student gives to the education and leisure criteria, we immediately see the effect on the global priorities of the alternatives. Thus, if we modify the importance that student gives to these two criteria we see how the choice of Erasmus destination would be different, considering the global priorities that are obtained in this case, as seen in Figure 7.5.
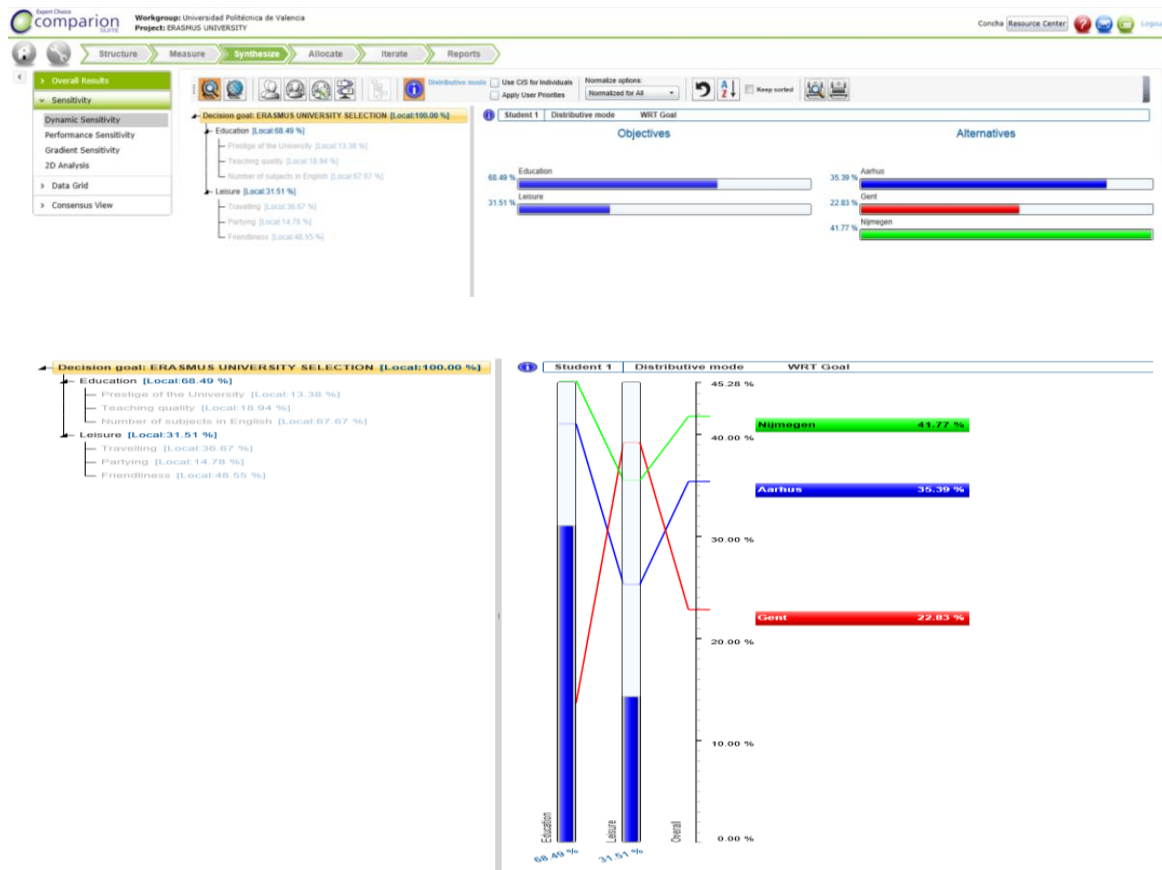
Figure 7.4. Results and sensitivity analysis graphs in *Expert Choice* for Erasmus destination example
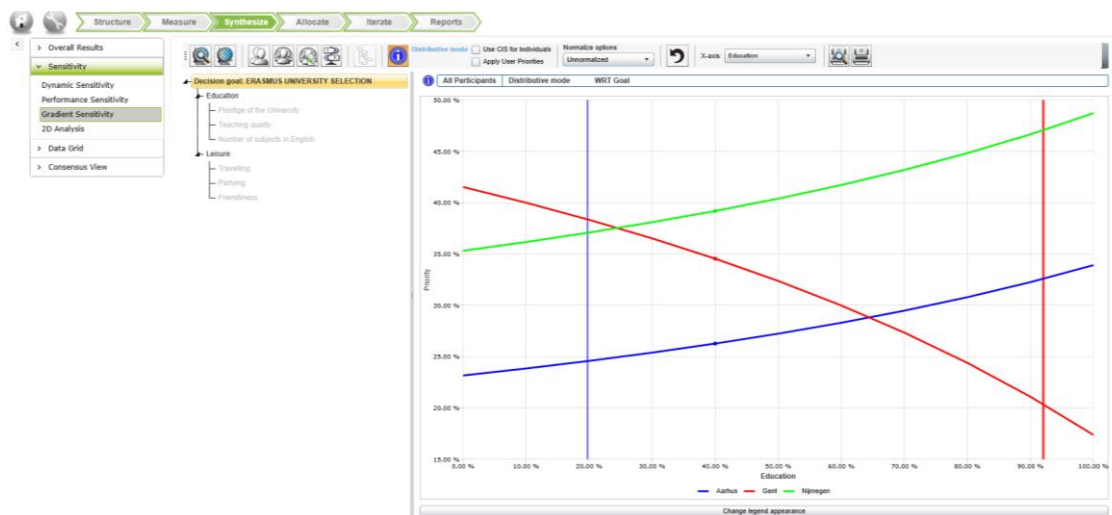

Figure 7.5. Sensitivity Analysis in *Expert Choice* Erasmus for destination example: modification of the weight of education

216

7.2. PROMETHEE METHOD

7.2.1. INTRODUCTION

Brans developed the PROMETHEE (Preference Ranking Organisation Methods for Enrichment Evaluations) in 1982 and its methodology has been successfully applied in finance, investment planning, industrial location, tourism, hospital management and water management among many other fields.

A discrete multicriteria problem is characterized by a finite set of **alternatives** *(a₁, a₂...aᵢ... aₙ)* and a set of evaluation **criteria** *(g₁, g₂...gᵢ... gₖ)*. We can be interested in maximizing some criteria and minimizing others. As we saw in the previous chapter, in general there is no alternative that optimizes all criteria. Suppose you wanted to buy a car. The criteria that may be important for you are the price, design, consumption, security, etc. No car optimizes all criteria at the same time: the cheapest does not imply the least consumption, better security and better design. Thus we need to choose the best compromise solution, which depends not only on the **basic** data, represented in an evaluation table (Table 7.6), but also on our individual preferences. Therefore, we need additional information to represent these preferences.

Table 7.6. Evaluation table

Alternatives	Evaluation criteria					
	g_1	g_2	…	g_j	…	g_k
a_1	$g_1(a_1)$	$g_2(a_1)$	…	$g_j(a_1)$	…	$g_k(a_1)$
a_2	$g_1(a_2)$	$g_2(a_2)$	…	$g_j(a_2)$	…	$g_k(a_2)$
…	…	…	…	…	…	…
a_i	$g_1(a_i)$	$g_2(a_i)$	…	$g_j(a_i)$	…	$g_k(a_i)$
...	…	…	…	…	…	…
a_n	$g_1(a_n)$	$g_2(a_n)$	…	$g_j(a_n)$	…	$g_k(a_n)$

Suppose we have a problem in which we have to prioritize five investments I1, I2 ... I5. The evaluation criteria are the Net Present Value (NPV), Internal Rate of Return (IRR), employment, sales and environmental impact. It is of interest to maximize the first four criteria and minimize the latter.

In Figure 7.6 we present the evaluation table for this problem, which was obtained with D-Sight, the software with which we have made the calculations and presented the results in the remaining figures in this section.

Figure 7.6. Evaluation table for prioritizing investments

Dominance relations associated with multicriteria problem are defined as follows:

For each pair of **alternatives** *a* and *b*, *a* **is preferred** to *b* (*aPb*) if for all considered criteria their value for *a* is equal to or better than their value for *b* and there is at least one criterion for an alternative *a* is better than *b*.

Two alternatives *a* and *b* are **indifferent** (*aIb*) if the value of all the criteria is the same for *a* and for *b*.

Two alternatives *a* and *b* are **incomparable** (*aRb*) if the value of some criterion is better for *a* than *b* and there is at least one criterion that is better for *b* than for *a*.

What dominance relations are there between alternatives in Figure 7.6? The alternatives that are not dominated by any other solutions are **efficient solutions**, a concept we saw in the previous chapter. In real problems many alternatives are incomparable, so we need **additional information** to make the decision. This information can be: *trade-offs* between criteria, weights that indicate the relative importance of criteria, a value function that adds all the criteria in a single function and therefore with a single criterion for which we can find the optimal preferences associated to each pairwise comparison within each criteria, thresholds for the boundaries of preferences, etc.

Many multiple criteria methods have been proposed that need a table like Table 7.6 and they differ in the additional information required. The purpose of the methods is to reduce the incomparability number. All methods should meet a number of requirements such as:

1. They should take into account the size of the difference in the behaviour of the alternatives for each objective: $d_j(a,b) = g_j(a) - g_j(b)$

2. Remove the **scale effect** as the valuation of the criteria is measured in units that may be different.

3. In pairwise comparisons, an appropriate method should inform if an alternative *a* is preferred to *b*, if it is indifferent or if it is incomparable.

4. As each multiple criteria method requires distinct information and calculation procedures, the solutions obtained can be different. Therefore, it is important that the decision makers understand the methods. It is also convenient to analyse the problem using various techniques to propose robust solutions.

5. Methods should provide information about the conflicts between criteria.

6. Most methods require the weights of the relative importance of the criteria. The weights can be assigned directly or using *AHP*. In any case it is desirable to have tools to do sensitivity analyses that allow us to see the impact of the weights in the solution.

Bernard Roy proposed building improvement relationships, enriching the dominance relationships which are based on *ELECTRE* (ELimination Et Choix Traduisant la REalité/ ELimination and Choice Expressing Reality). *PROMETHEE* belongs to a group of multiple criteria methods, known as *Outranking* methods. In both cases there are several versions of the method, depending on the type of problem to be solved (*PROMETHEE I, II, III, IV, V, VI*).

7.2.2. INFORMATION FOR PREFERENCES MODELLING

In addition to an evaluation table, *PROMETHEE* requires information on the weights of the relative importance of the criteria, we call these $w_1, w_2 \dots w_k$. The sum of all is the unit:

$$\sum_{j=1}^{k} w_j = 1$$

The preference structure of *PROMETHEE* is based on pairwise comparisons, as in *AHP*. However, in this case comparisons are based on the difference between the valuations of two alternatives of a particular criterion. The larger the difference between evaluations of the alternatives, the greater is the preference for the alternative which behaves better. When the difference is small, the decision maker can consider it to be negligible. These preferences are represented by real numbers between 0 and 1. Figure 7.6 shows that investment I5 is 50 units better than I4 and 150 units better than I1 for NPV.

219

In general, for each criterion j, the decision maker has the preference function between *a* and *b*, $P_j(a,b)$, a function that depends on the difference between the behaviour of the alternatives:

$P_j(a,b) = F_j[d_j(a,b)]$ for all alternatives of the problem, where

$d_j(a,b) = g_j(a) - g_j(b)$ and

$0 \le P_j(a,b) \le 1$

In case the criterion is maximized, the preference will be *a* to *b* for criterion evaluations and the preference can have the shape of Figure 7.7. When the deviation is negative, the preference is 0, and therefore

if $P_j(a,b) > 0$ then $P_j(b,a) = 0$

When we want to minimize the criterion it would be the other way round or we consider the preference function as follows:
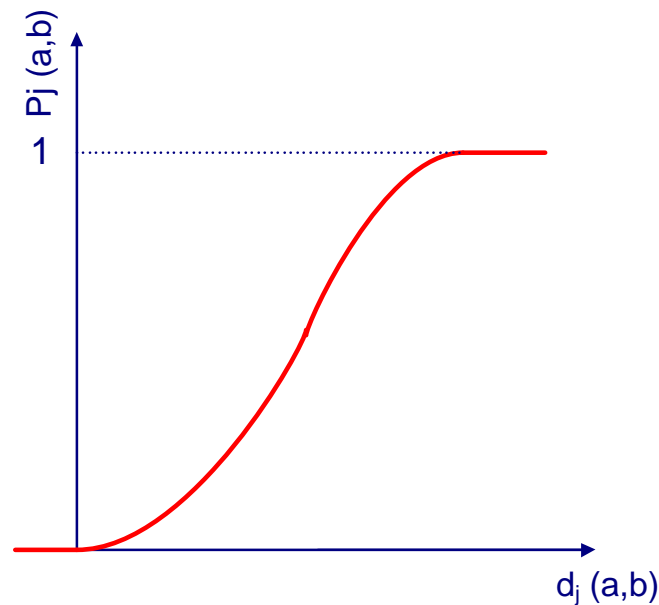
$P_j(a,b) = F_j[-d_j(a, b)]$

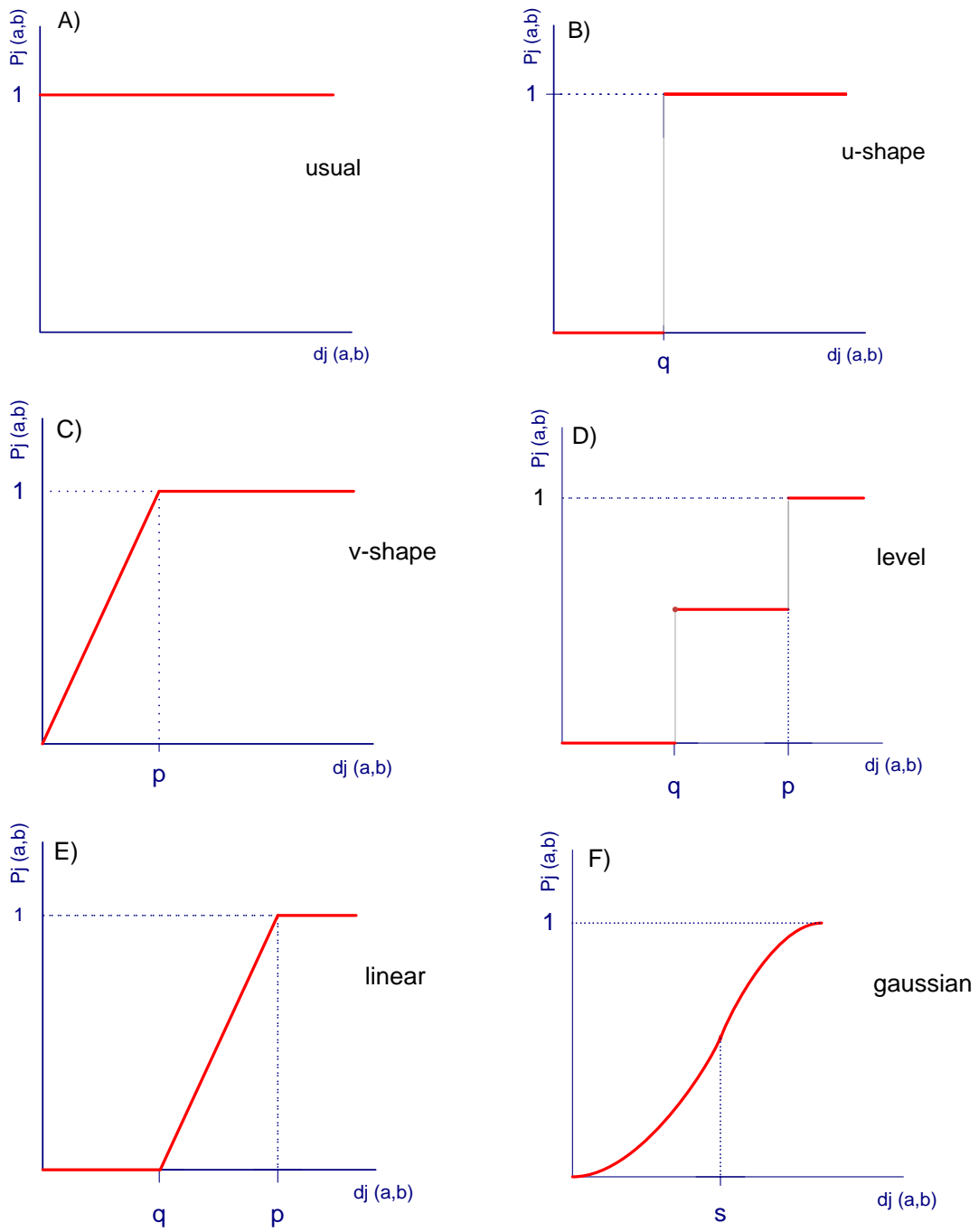**Figure 7.7. Preference function**

Figure 7.8. D-Sight Preference Functions

For each criterion, we must propose a preference function. The most common functions are those presented in Figure 7.8, and they are the ones that you can choose from the D-Sight software. But in the former case, *usual* (A), where the preference is 1 if the difference between *a* and *b* is positive, and zero otherwise. In other cases we have to define one or two parameters. We call **q** the **indifference threshold**, **p** the **strict preference threshold** and **s** is an intermediate value between them. q is the value of the largest deviation that the decision maker considers negligible, while the preference threshold p is the smallest value of the deviation between alternatives to be considered sufficient for a strict preference of one alternative to another. In case B parameter q must be set, which is p in case C and, in cases D and E the two parameters must be set, the indifference threshold and strict preference threshold. And in the Gaussian preference function (F) s must be set, which will be between p and q. In all cases, when the difference in the behaviour of the alternatives is negative, the preference will be zero (Figure 7.8).

Figure 7.9 shows that we have chosen a type of linear preference function for all objectives, indifference thresholds and preference for each case and the weights of the criteria.

Evaluations	Alternatives	Criteria	Parameters ⤢ ✕	Hierarchy						
Criteria	Type	Min/Max	Function	Abs/Rel	Indiff.	Pref.	Weight	Unit	Scale	Decimals
NPV	Pair Wise	Maximize	Linear	Absolute	0	50	25,0%		Numerical	3
IRR	Pair Wise	Maximize	Linear	Absolute	0	5	25,0%		Numerical	3
Employment	Pair Wise	Maximize	Linear	Absolute	0	3	20,0%		Numerical	3
Sales	Pair Wise	Maximize	Linear	Absolute	0	15	10,0%		Numerical	3
Environmental impact	Pair Wise	Minimize	Linear	Absolute	0	100	20,0%		Numerical	3

Figure 7.9. Preference functions and parameters

7.2.3. PROMETHEE I AND II

To apply the method we need to know the evaluation table, the weights of the criteria and the preference functions. First we define the **Aggregated Preference Indices** for each pair of alternatives *a* and *b*:

$$\pi\,(a,b) = \sum_{j=1}^{k} Pj(a,b)w_j$$

$$\pi\,(b,a) = \sum_{j=1}^{k} Pj(b,a)w_j$$

Where **π (a, b)** expresses **the degree to which the alternative *a* is preferred over *b*** and π *(b, a)* the degree to which the alternative *b* is preferred over *a*. In most cases there are criteria for which *a* is preferred to *b* and others for which b is preferred to the alternative a. These indices have the following properties:

π *(a, a) = 0*

$0 \leq \pi$ *(a, b)* $\leq$ *1*

$0 \leq \pi$ *(b, a)* $\leq$ *1*

$0 \leq \pi$ *(a, b) + π (b, a)* $\leq$ *1*

If the aggregated preference index of *a* over *b*, $\pi(a, b)$ is close to zero, then there is a weak global preference for *a* over *b*, and if it is close to 1, there is a strong global preference. Figure 7.10 shows the aggregated preference indices for the investments example.

Thus, using the data in Figures 7.6 and 7.9 (weights of objectives and preference functions) we obtain

π *(I5, I2) = $\sum_{j=1}^{k} Pj(I5, I2)w_j$= 1*0.25+0*0.25+1*0.2+1*0.1+0*0.2 = 0.55*

π *(I4, I2) = 0*0.25 + 1*0.25 + 1*0.20 + 0.666*0.10 + 0*0.20 = 0.5166*

Calculate the aggregated indices for other pairs of investments. Check the results in Figure 7.10. Analyse the preferences of some investments over others as well.

As each alternative is compared with the other (n-1) positive and negative outranking flows are defined. **Positive outranking flow** expresses to what extent an alternative outranks all the others. The higher the positive outranking flow, the better the alternative. The value of this expresses the strength of the alternative.

$$\varphi^{+}(a) = \frac{1}{n-1} \sum_{x \in A} \pi(a, x)$$

In contrast, the **negative outranking flow** indicates to what extent an alternative is overcome by the other. It is therefore an indicator of weakness and the alternative is better when its negative flow is smaller.

$$\varphi^{-}(a) = \frac{1}{n-1} \sum_{x \in A} \pi(x, a)$$

For the investment example

$$\varphi^{+}(I1) = \frac{1}{4}(0.2 + 0.3 + 0.2 + 0.2) = 0.225$$

$$\varphi^{-}(I1) = \frac{1}{4}(0.6 + 0.3 + 0.8 + 0.8) = 0.625$$

Calculate positive and negative flows of other investments. Check the results in Figure 7.11.

Pref. Degree Table ⟗ ✕

	I1	I2	I3	I4	I5
I1	0,000	0,200	0,300	0,200	0,200
I2	0,600	0,000	0,800	0,200	0,200
I3	0,300	0,200	0,000	0,200	0,200
I4	0,800	0,517	0,800	0,000	0,650
I5	0,800	0,550	0,800	0,350	0,000

Figure 7.10. Aggregated preference indices between alternatives

In the **PROMETHEE I** we obtain the partial ranking P^I, I^I and R^I of the alternatives from the **positive and negative flows**. We do not usually get the same rankings from the two kinds of flows.

Alternative a is preferred to b, a P^I b, if a positive flows are greater than b, and the lower negative or the positive of a and b are equal, or a lower negative or a positive flows are higher than b and the negatives are equal in both alternatives.

Alternative a is indifferent to b, a I^I b if both positive and negative flows are equal in a and b.

Alternative a is incomparable to b, a R^I b, if a positive flows are greater than b and the negative also or a positive flows are lower than b and the negative are also lower than b. This usually happens when the alternative a is good for a group of criteria where b is weak and the alternative b is good for criteria in which a is not.

Net Flow Table ⟗ ✕

Alternative	Rank	Score	Positive Flow (Pairwise)	Negative Flow (Pairwise)
I1	4	-0,400	0,225	0,625
I2	3	0,083	0,450	0,367
I3	5	-0,450	0,225	0,675
I4	1	0,454	0,692	0,238
I5	2	0,312	0,625	0,312

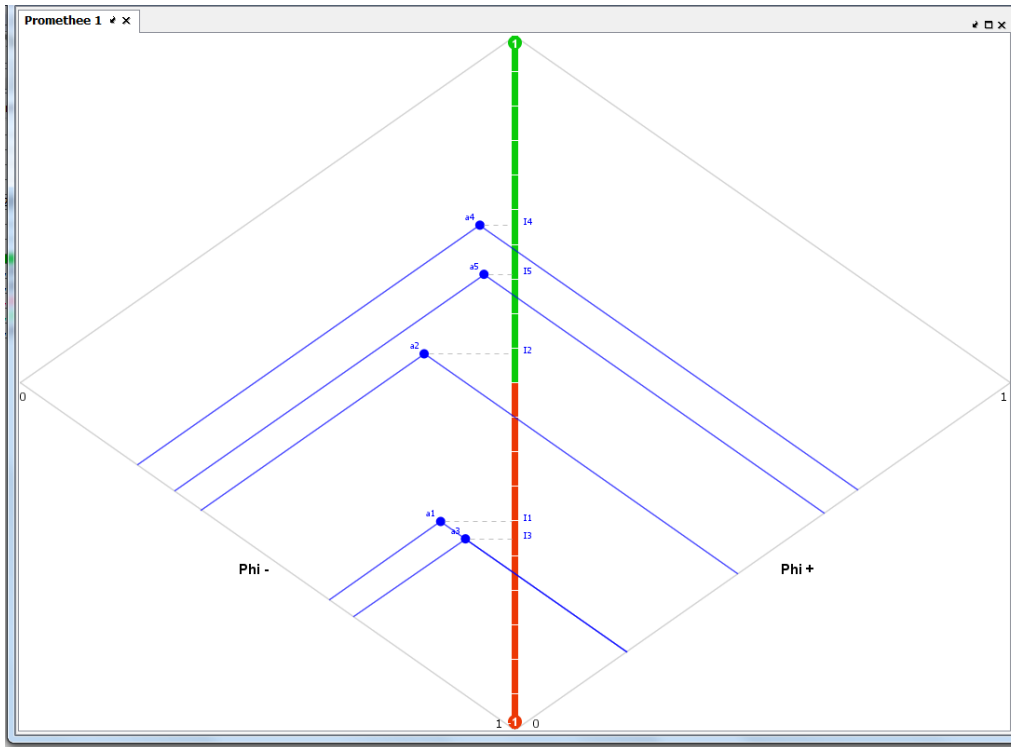**Figure 7.11. Positive, negative and net flows of the alternatives**

Figure 7.12. Representation of the partial ranking (positive and negative flows)
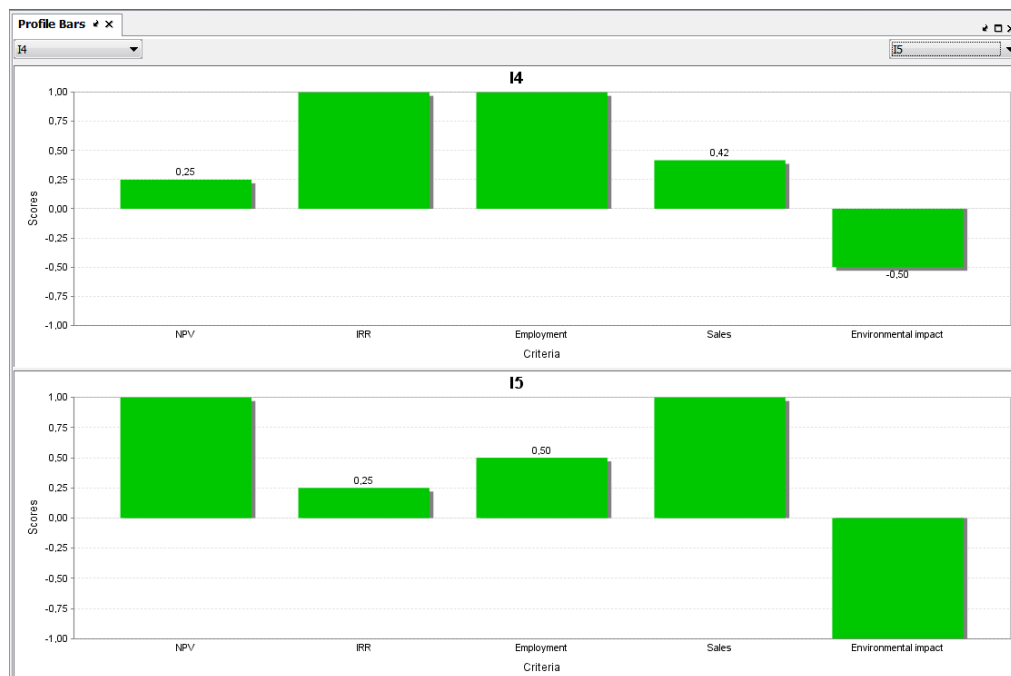

Figure 7.13. Profile of alternatives Investment 4 (I4) and Investment 5 (I5)

In Figure 7.14 we have the matrix of net flows for all alternatives of the decision problem that is obtained when the whole weight (100%) is given to a criterion.

	NPV	IRR	Employment	Sales	Environmental impact
I1	-0,750	-1,000	-0,250	-0,500	0,688
I2	0,250	0,250	-0,250	0,083	0,000
I3	-0,750	-0,500	-1,000	-1,000	0,812
I4	0,250	1,000	1,000	0,417	-0,500
I5	1,000	0,250	0,500	1,000	-1,000

Figure 7.14. Net flows matrix

The PROMETHEE II provides a complete ranking of alternatives (**P^{II}, I^{II}**). In this case we define net flow of alternative *a* as the balance between positive and negative flows:

$$\varphi\ (a) = \varphi^+\ (a) - \varphi^-\ (a)$$

The higher the **net flow,** the better the alternative and therefore, in this method, **all alternatives are comparable**. The values of net flows of the alternatives are between -1 and 1 and the sum of all of them is 0. If the net flow of **alternative *a*** is **positive**, it is better than all alternatives for all criteria and when the net flow is negative then it is worse than the other alternatives. See in Figure 7.11 the values of net flows in score column for investment example.

PROMETHEE II is easy to use, but the incomparability analysis can help us make decisions in real problems. As the net flow gives us full ranking, it may be compared with a utility function. Figure 7.15 shows the net flow of each investment, which provides us with the ranking of the alternatives.
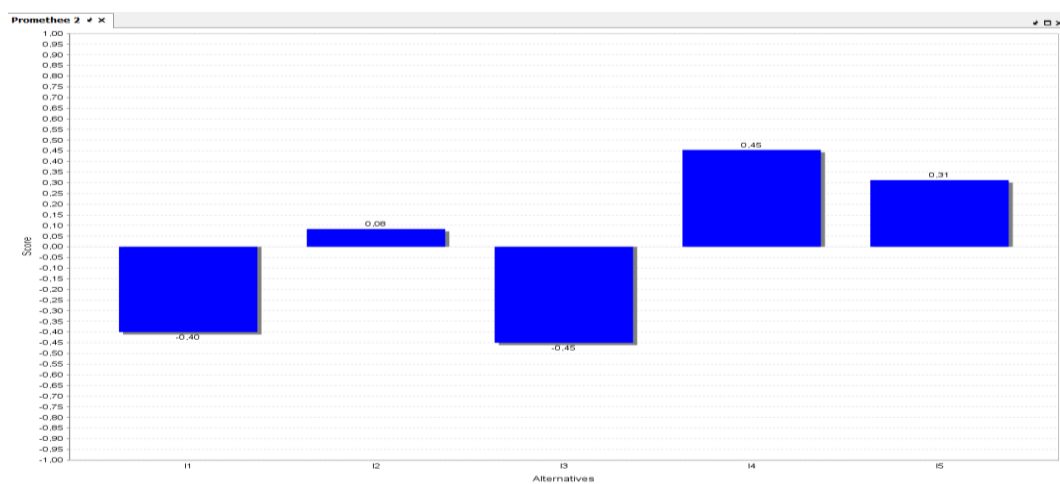

Figure 7.15. Net flows of investment alternatives

The net flow of the alternative *a* for criterion j, φ_j **(a),** is obtained when only this criterion is considered, i.e. 100% of the weight is given to that criterion. This concept expresses how this alternative outranks the other alternatives for this criterion (φ_j **(a) > 0**) or how it is outranked by the other alternatives for criterion j (φ_j **(a) < 0**). The profiles of the alternatives show their quality in all considered criteria. Figure 7.13 presents profiles for I4 and I5 investments. We can see how alternative I5 has the greater NPV and sales than I4; however the negative environmental impact is greater in I5 than in I4.

The net flow of an alternative is the scalar product between the vector of weights and the profile vector of this alternative. This property is used to construct the GAIA plane, which is a graphical tool for analysing multiple criteria problems (Global Visual Analysis, **GVA**) and can be seen in Figure 7.16. The alternatives are represented as points and the criteria are the axes. The red axis is the decision. The GAIA plane shows the discriminating power of the criteria, the conflicting aspects and the quality of the alternatives on different criteria. This plane has a number of interesting properties to interpret the results. For example, the longer the axis of a criterion the more discriminant is that criterion. The criteria with similar preferences have axes oriented approximately in the same direction. The conflict criteria are oriented in opposite directions. The criteria that are not related to others in terms of preferences are represented by orthogonal axes. Similar alternatives are represented by close points and good alternatives with regard to a criterion are represented by points located in the direction of the axis of this criterion (see Figure 7.16).
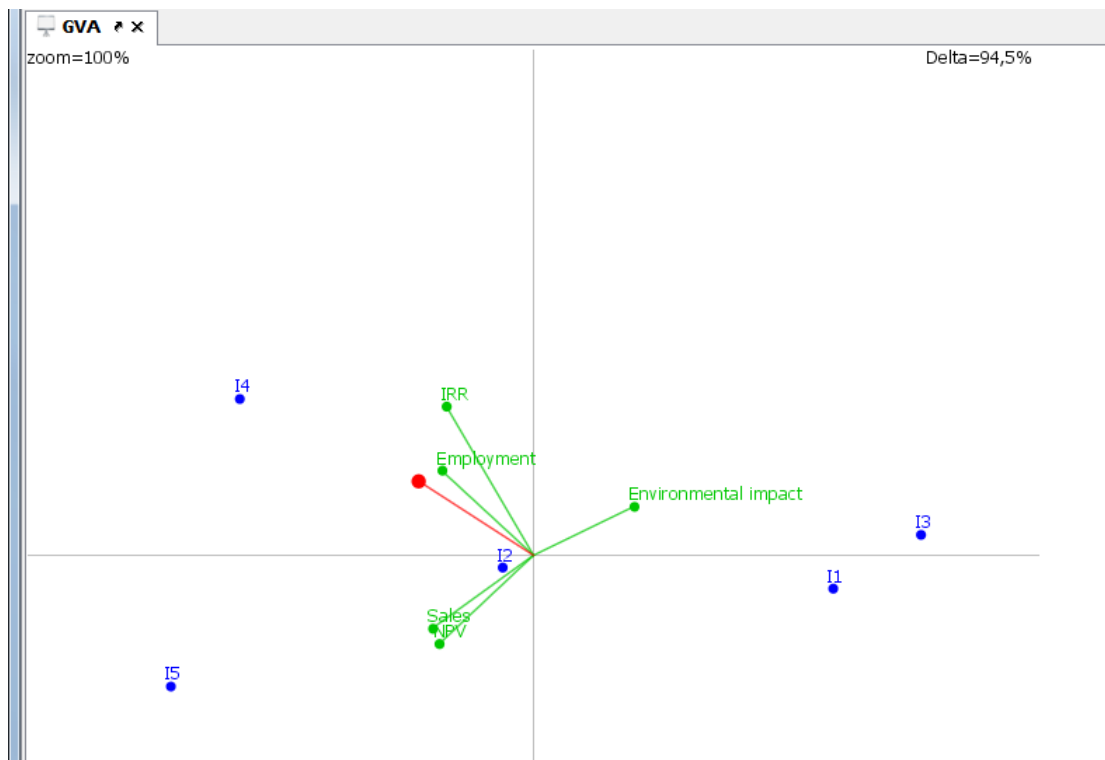

Figure 7.16. GAIA drawing

The D-Sight software provides a variety of graphics that we can use as a tool for analysis and presentation of results as Figures 7.17 and 7.18. The *spider web* graph represents multiple profiles of alternatives, criterion by criterion. The centre of the graph represents the value -1 and the end point of the axes represents +1 (Figure 7.17). It also allows sensitivity analysis such as that in Figure 7.19, which shows that the best choice is insensitive to the weights of the criteria.
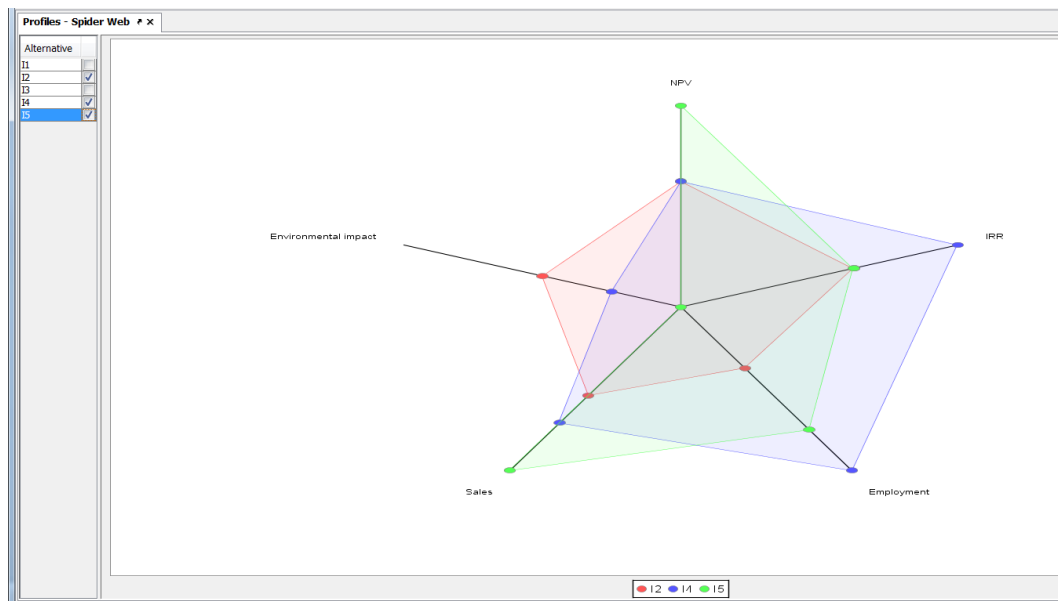

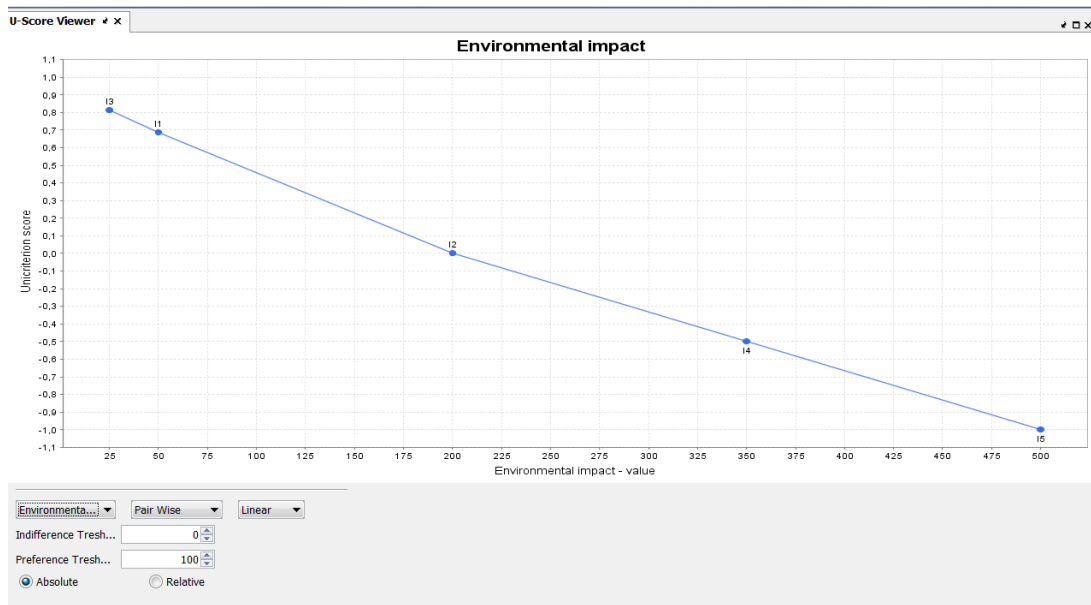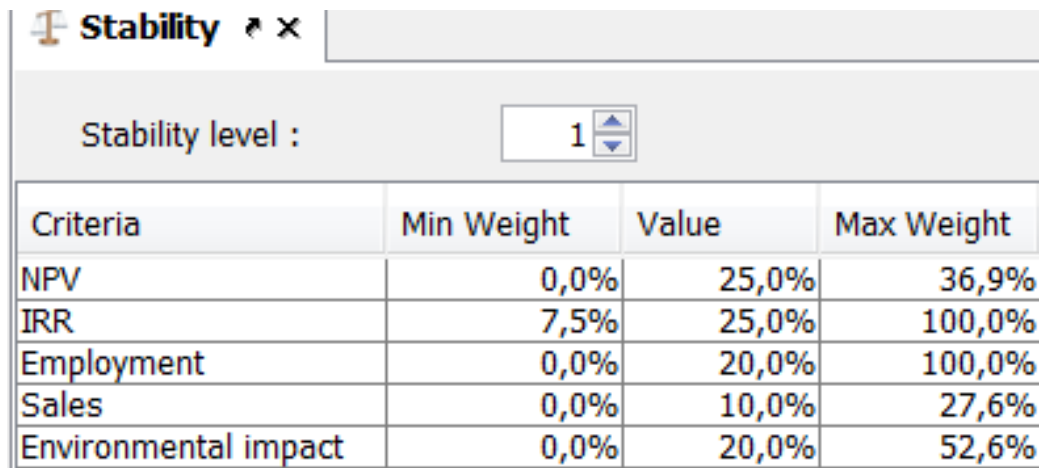**Figure 7.17. Spider web graph of the selected alternatives (I2, I4 e I5)**

Figure 7.18. Graphical representation of the alternatives for the environmental impact

Figure 7.19. Sensitivity analysis

7.3. COLLABORATIVE DECISION MAKING

We can also apply AHP when the decision is made in a group. In this case we can distinguish two fundamental aspects. Firstly, how to add individual preferences into a collective judgment and secondly, how to build a group preference from individual preferences.

If a group of people are involved in making decisions, we have to determine consensus matrices and local and global weights or priorities that represent the preferences and priorities of the group. We can use different techniques to obtain them. One of the most used technique is **the geometric mean of all pairwise comparisons**, as defended Saaty and others (Saaty and Peniwati, 2008; Xu, 2000). Saaty considers that the geometric mean is necessary because of comparison between two elements j and i, a_{ji}, should give us the reciprocal value assigned to a_{ij} the original comparison. The arithmetic mean does not satisfy this reciprocal relationship. Moreover, given a group of inconsistent individuals, the inconsistency of the group by adding individual judgments by the geometric mean is at most equal to the largest individual inconsistency. Another alternative to add preferences or judgments and to obtain consensus matrices in group decision making is to use the goal programming models developed by González-Pachón and Romero (2004, 2007).

Both AHP and goal programming models are suitable techniques for group decision making. However AHP has several advantages. The first is the simplicity of the method, easily understandable by all members of the group who do not need to be experts in decision making techniques. The second advantage is the use of the geometric mean for adding preferences/judgments of pairwise comparison, where we always obtain consistent consensus matrices from consistent individual matrices. However, one of the most frequent criticisms of AHP is that when adding a new option, the ranking can change (rank reversal). Saaty usually argues by saying that it would be a different decision problem.

The goal programming models developed by González-Pachón and Romero have an advantage compared with AHP, they do not require that preferences or individual judgments are consistent. However, these models have the drawback that the consensus matrix obtained by adding preferences can be inconsistent in theory. It is possible to develop a model to avoid this, but a nonlinear goal programming model would have to be solve. Also note that the method is difficult for participants without previous training in optimization techniques to understand.

In Annex 3 we explained, with an example, how to use the *Expert Choice Comparion Core* software for collaborative decision making that allows us to aggregate the preferences of a working group and schedule sessions for making collective decisions in real time. Finally, note that we can also use *PROMETHEE* for collaborative decision making. For this purpose there is D-Sight software web platform.

7.4. SUMMARY

In this chapter we have seen two widely used methods to solve multiple criteria decision problems when we have a small number of decision alternatives. *AHP* is based on designing a decision hierarchy that represents the problem and allows the weights of objectives and priorities of alternatives from the pairwise comparison of elements to be obtained. In this case, the decision-maker expresses value judgments about the importance or preference of one element over another, using the Saaty scale. The judgments must be consistent in order to select or prioritize the alternatives. *Expert Choice* software and *Super Decision* help address the multiple criteria decision problems, both individually and at the group level. *PROMETHEE* is another method that is also based on pairwise comparison of alternatives. However, in this case it is necessary to know the behaviour of the alternatives for each criterion quantitatively or qualitatively. In this method the preferences of the decision maker are incorporated through preference functions. This approach compares the alternatives by using the net flow, which is an independent concept of the units in which the criteria are measured. D-Sight is a tool for solving discrete multiple criteria problems with the PROMETHEE method for individual and collaborative decision making using a web platform.

7.5. SELECTED REFERENCES

1. Belton, V. and Stewart, T.J. 2003. Multiple Criteria Decision Analysis— an integrated approach. Kluwer Academic Publishers, Dordrecht, The Netherlands

2. Brans, J.P. and Mareschal, B. 2005. PROMETHEE Methods, chapter 5 en Multiple Criteria Decision Analysis. State of the Art Surveys. Edited by Figuera, J.; Greco, S. y Ehrgott, M. Springer. Pp.163-195.

3. Expert Choice Comparion Core.2013. www.expertchoice.com.

4. D-Sight. 2013. http://www.d-sight.com.

5. González-Pachón, J. and Romero, C. 2004. A method for dealing with inconsistencies in pairwise comparisons, European Journal of Operational Research, 158 (2), 351–361

6. González-Pachón, J. and Romero, C. 2007. Inferring consensus weights from pairwise comparison matrices without suitable properties, Annals of Operational Research, 154 (1), 123–132.

7. Saaty, T.L. and L.G. Vargas 2001. Models, methods, concepts & applications of the Analytic Hierarchy Process. Kluwer Academic Publishers.333pp.

8. Saaty, T.L. 2006. Fundamentals of decision making and priority theory with the analytic hierarchy process. RWS Publications, Pittsburgh, USA. 478 pp.

9. Saaty, T.L. 2008. Decision making with the analytic hierarchy process, Int. J. Services Sciences, 1 (1), 83-98.

10. Saaty, T.L. and Peniwati, K. 2008. Group Decision Making: Drawing out and Reconciling Differences. RWS Publications. 385 pp.

11. Super Decisions. 2012. http://www.superdecisions.com.

7.6. CASE STUDIES

CASE STUDY 1: SELECTION OF ERASMUS/LEONARDO DESTINATION

Each student must design a hierarchy that represents the personal criteria and objectives to consider in the destination selection where you would attend a year or a semester at another university, either within the Erasmus, Leonardo or any other programme of your interest. The case study can be done with *Excel* and/or *Expert Choice*.

1. Carry out the hierarchy that represents the criteria and objectives you would consider at your personal level.
2. Obtain the matrices of pairwise comparisons.
3. Calculate local and global priorities.
4. Carry out the sensitivity analysis.
5. Write a brief report to support the choice.
6. Choose a destination for a group of friends through collaborative decision making.

CASE STUDY 2: PREFERENCE AGGREGATION AND CALCULATION OF PRIORITIES THROUGH GOAL PROGRAMMING

González-Pachón and Romero (2007) have developed a goal programming model for obtaining the consensus matrix, for example for a group of students that choose the same Erasmus destination.

Decision variables of the model that allows us to obtain the consensus matrix for a group are:

$R_{ij}^{C}=$ the consensus value ratio that quantifies the aggregated judgment when comparing the criterion/alternative i with j.

N_{ij}^{K} *and* $P_{ij}^{K}=$ positive and negative deviation variables from the goal when the student K compares the criterion/alternative i with j

Upper and lower limits of consensus ratios due to the application of the fundamental Saaty scale:

$$0.111 \leq R_{ij}^{C} \leq 9 \qquad\qquad i,j = 1,2,\dots,n$$

In the case that the consensus ratio between two criteria and the value of ratio of the student K are different, this difference will be positive or negative deviation variables as indicated by the model goals.

$$R_{ij}^C + N_{ij}^K = R_{ij}^K + P_{ij}^K \qquad i,j = 1,2, \dots, n \quad i \neq j \qquad K = 1,2, \dots, m$$

The achievement function is to minimize the sum of the deviation variables for all students in the group.

$$MIN \sum \sum \sum \left(N_{ij}^K + P_{ij}^K\right) \quad i,j = 1,2, \dots, n \quad i \neq j \qquad K = 1,2, \dots, m$$

In a second phase we obtained the weights of relative importance that the group of students i give to the criterion r from the consensus matrix obtained with the previous model, using another goal programming model also developed by González-Pachón and Romero (2004).

In the latter model $\boldsymbol{W_{ri}}$ **decision variables** are the weights of the student group i for criterion r. In the same way as for the previous model, we define the variables of positive and negative deviation for goals and student group i (i = 1, 2... m).

$$W_r^i + N_{r,s}^i = R_{r,s}^{Ci} \cdot W_{si} + P_{r,s}^i \qquad r,s = 1,2, \dots, n \quad r \neq s \qquad i = 1,2, \dots, m$$

It is also necessary to add the constraint that the sum of all weights is 1 for each group of students i:

$$\sum_{r=1}^{r=n} W_r^i = 1 \qquad r = 1,2, \dots, n$$

The achievement function is to minimize the sum of all positive and negative deviation variables for all criteria.

$$MIN \sum \sum \sum \left(N_{r,s}^i + P_{r,s}^i\right) \qquad r,s = 1,2, \dots, n \quad r \neq s$$

1. Develop the goal programming models that address the problem of group decision making of the case study 1 and obtain the solution.
2. Compare the results obtained from applying the AHP method.

CASE STUDY 3: CAR SELECTION

Develop a multiple criteria model that allows you to choose which car to buy from the five models on the market that are the most interesting to you. Please consider at least the criteria of price, consumption, pollution and design.

1. Develop a model, obtain the necessary data and the solution by the *AHP* method.

2. Solve the problem by using the *PROMETHEE* method. Write a report comparing the data required and the solution obtained with those from the *AHP* method.

CASE STUDY 4: PHONE SELECTION

Develop a multiple criteria model that allows you to prioritise which phone to buy taking into account several criteria. Obtain the necessary data and the solution by using *AHP* and PROMETHEE methods. Finally, write a report comparing the data required and the solution obtained with both methods.